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CLASSICAL DYNAMICS OF CHARGED GRAVITATIONAL BINARY SYSTEMS FROM SCATTERING AMPLITUDES

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Resumen

Se estudia la dinámica clásica de un sistema de dos cuerpos compactos no rotatorios que interactúan gravitacional y electromagnéticamente. La dinámica clásica está codificada en la amplitud de scattering de dos a dos partículas con carga eléctrica, masivas y de espín cero. Se modela el sistema de dos cuerpos por dos campos escalares complejos acoplados a la teoría de Einstein-Maxwell. Luego, se calcula la amplitud de scattering a nivel árbol y a un lazo y se extrae la física clásica utilizando herramientas de teoría de campo efectivo. Nuestro resultado central es el Hamiltoniano efectivo de dos cuerpos para objetos compactos no rotatorios en el sector conservativo a segundo orden en la constante de Newton y la constante de acoplamiento electromagnética. Adicionalmente, se calcula el ángulo de dispersión, un observable clásico, de los dos cuerpos masivos en un escenario de dispersión.

Classical Dynamics of Charged Gravitational Binary Systems from Scattering Amplitudes

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Abstract: We study the classical dynamics of a system of two spinless compact bodies interacting gravitationally and electromagnetically. The classical dynamics are encoded in the two-to-two scattering amplitude of electrically charged, massive spin-zero particles. Our setup is to model the two-body system by two complex scalar fields coupled to Einstein-Maxwell theory. We then compute the four-point scattering amplitude up to the one-loop order and extract the classical physics using effective field theory tools. Our central result is the conservative two-body effective Hamiltonian for spinless compact objects at second order in Newton's constant and the electromagnetic coupling. Additionally, we compute the scattering angle, a classical observable, of the two massive bodies in the scattering scenario.

Contents

1 Introduction

The existence of gravitational waves (GWs) is one of the central predictions of the theory of general relativity (GR) [1]. The first indirect evidence of gravitational waves came from the observation of the decreasing orbital period of the Hulse-Taylor binary pulsar [2, 3], in accordance with Einstein's quadrupole formula [4]. It was not until nearly a century after Einstein's prediction that a direct detection of gravitational waves was reported by the LIGO and Virgo collaborations [5], ushering in the era of gravitational-wave science. The sources of the gravitational waves that have been detected so far by Earth-based observatories are inspiral binary systems of compact objects. The first observed gravitational-wave event corresponded to a Binary Black Hole (BBH) system. Gravitational-wave events for a Binary Neutron Star (BNS) system [6] and a Neutron Star-Black Hole Binary (NSBH) system [7] have also been observed.

Since the first observing run (O1) of the LIGO detectors, additional upgrades on the detectors have been performed, thereby increasing their sensitivity and performance [8]. With the KAGRA detector [9] joining in on the current observing run (O4), the LIGO-Virgo-KAGRA (LVK) collaboration has announced more binary event candidates in O4 than in the previous three runs combined [8]. Alongside the increased precision of upcoming LVK runs, the coming years will bring a new generation of ground-based detectors such as Cosmic Explorer [10], Einstein Telescope [11] and LIGO-India [12], as well as the space-based Laser Interferometer Space Antenna (LISA) detector [13]. These future detectors will undoubtedly make gravitational-wave science a very promising and active research field as they open to us a new window into the Universe never before probed.

What exactly can we infer from gravitational-wave signals? We list a few of the insights into fundamental physics that gravitational-wave observations can provide:

Tests of GR. GW observations are made through waveform models which are built solving Einstein's field equations. Observations of BBH mergers in particular offer the ideal testing ground for GR in the strong-field regime [14, 15]. Since waveform models can also be built in modified theories of gravity, GW observations also allow us to test deviations from GR [16, 17].

Equation of state of neutron stars. The physics regarding the properties of the core of neutron stars is still not well understood. Future GW observatories will be able to probe the densities and pressures in neutron stars [18], as well as give constraints on their equations of state [19, 20].

Properties of dark matter. The nature of dark matter remains a mystery in astrophysics and cosmology. Alterations in the orbital dynamics of compact binaries would leave an imprint in their GWs, providing indirect evidence on the properties of dark matter [21]. Additionally, primordial black holes have been proposed as candidates of *some* of the dark matter as result of LIGO-Virgo detections [22].

Dark energy. To account for the accelerated expansion of the universe, dark energy is introduced as a cosmological constant in the so-called ΛCDM model, the standard model of cosmology. The nature of dark energy, however, is still an open problem. Measurements of the luminosity distance to a binary merger can be extracted from gravitational waveforms. If their redshifts are also obtained from EM observations, inferences can be made on dark energy equation-of-state, dark matter and dark energy density parameters [23].

As mentioned before, the gravitational-wave signals observed by LVK detectors come from coalescences of compact objects. The temporal evolution of these binary systems can be divided into three phases as illustrated in Fig. 1. The earliest stage is the *inspiral* phase. In this phase the two objects are orbiting each other in quasi-periodic motion while radiating gravitational waves; since the emitted gravitational waves carry away some of the system's energy, the two objects gradually come closer together. The gravitational waveform for this stage is nearly sinusoidal. Next is the

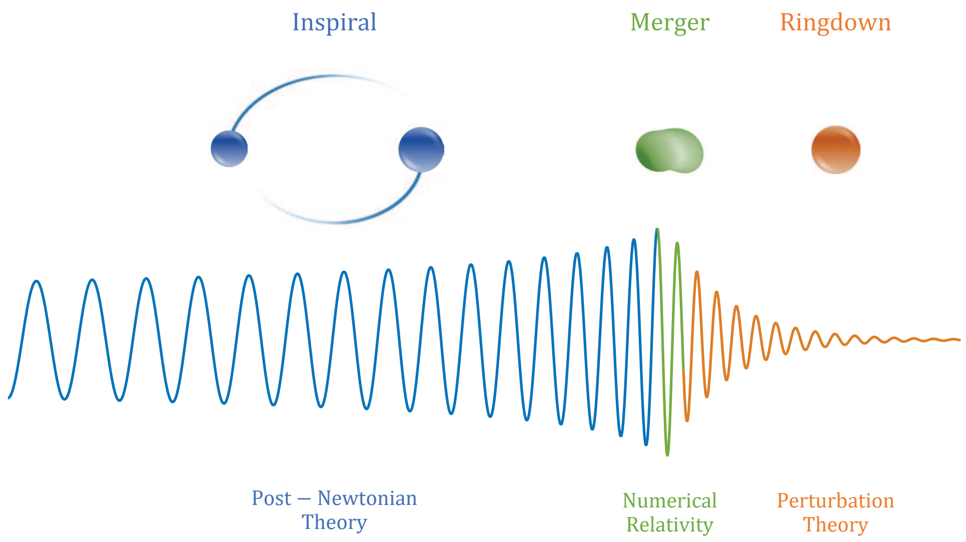


Figure 1. The three phases of a binary merger. Figure reproduced from Ref. [24].

merger phase. Here the separation between the objects approaches the characteristic size of the constituents and the two objects collide and fuse into one. The dynamics of this stage is nonperturbative and numerical methods have to be used to study it. The final stage is known as the *ringdown*. The gravitational waveform in this phase decays exponentially. For a BBH system, the merged remnant is a Kerr black hole which radiates gravitational waves through quasi-normal modes [25, 26]. Unlike the merger phase, the inspiral and ringdown phases can be treated using perturbative methods.

Future precision gravitational-wave detectors require high-precision theoretical models of the dynamics of compact binary mergers which are encoded in gravitational-waveform templates [27]. To construct accurate waveform templates there is input from a variety of complementary approaches, including the effective one-body (EOB) formalism [28, 29], the gravitational self-force (GSF) formalism [30, 31], the nonrelativistic general relativity (NRGR) formalism [32], numerical relativity (NR) [33–35], black hole perturbation theory (BHPT) [36], and the post-Newtonian (PN) [37–42] and post-Minkowskian approximations [43–51]. The upshot of the close synergy between these methods is the expected production of state-of-the-art predictions for the dynamics of compact binary systems and which capture the effects of spin, tidal deformations, and dissipative dynamics.

In principle, waveform templates can be produced entirely by solving Einstein's field equations numerically, since the exact solution to the two-body problem in GR remains unknown. However, this is not a viable option since simulations in NR are computationally expensive. As a result, complementary methods have to be considered. Approximate analytical solutions to Einstein's equations can be obtained using perturbation theory, solving the field equations order by order in a series expansion for one or more small parameters. Here we focus on the inspiral phase.

The first, and historically traditional, approach is the post-Newtonian approximation. This corresponds to the regime of small velocities, $v^2/c^2 \ll 1$, and weak gravitational field, $GM/rc^2 \ll 1$. Here, v and r are the relative velocity and position between the objects, and GM/c^2 is roughly the Schwarzschild radius of each object. Therefore, the PN expansion is a double expansion in two small parameters, but these are related to each other in bound orbits through the virial theorem, $v^2/c^2 \sim GM/rc^2$. At nPN order we have $n+1$ terms, which in powers of G and v^2 go as $G^{n+1}v^0, G^n v^2, \cdots, G^1 v^{2n}$. The current state-of-the-art for the conservative two-body potential is 6PN [52–54].

A second approach is given by the post-Minkowskian approximation. This involves an expansion in the weak field regime alone (in powers of GM/rc^2), and so is valid to all orders in velocity. The PM expansion includes the PN expansion in the sense that the nPM correction generates all PN terms at fixed order in G^n once expanded in powers of v^2. This approach is actually more suitable for the *scattering* scenario of the two-body problem, i.e., unbounded or hyperbolic orbits. Even though no gravitational-wave observation has been made for hyperbolic encounters, prospects for future LVK runs are encouraging [55]. Additionally, results from the scattering problem can be translated to the bound-state problem via analytic continuation of gravitational observables [56–58], or by obtaining an effective potential through effective field theory (EFT) matching [59] or by using a relativistic Lippmann-Schwinger equation [60]. Of particular note is the fact that considerable progress has been achieved in the PM framework by approaching the two-body problem through quantum scattering amplitudes [59, 61–65]. This success has been made possible by leveraging modern amplitudes tools that were developed for collider physics, including spinor-helicity formalism (see e.g. [66, 67]), double copy [68–70], generalized unitarity [71–73], and incorporating powerful integration procedures [74–77]. The current state-of-the-art for the conservative two-body potential is 4PM [64, 65], with results for the scattering angle and impulse partially known at 5PM [78].

The third approach is the gravitational self-force approximation. This is an expansion in powers of the mass ratio of the binary, $m_2/m_1 \ll 1$, that is, where the mass of one compact object is much bigger than the mass of the other. The two-body problem then reduces to solving for the geodesic motion of the lighter object in the exact spacetime of the heavier object (e.g., Schwarzschild spacetime) and then adding post-geodesic corrections in the mass ratio. Since the only small parameter is the ratio of the masses of the system, results can be obtained to all orders in G and so is valid in the strong-field regime. The computation of gravitational waveforms for the case of nonspinning compact binaries in quasi-circular inspiral orbit at second order in GSF was given in Ref. [79].

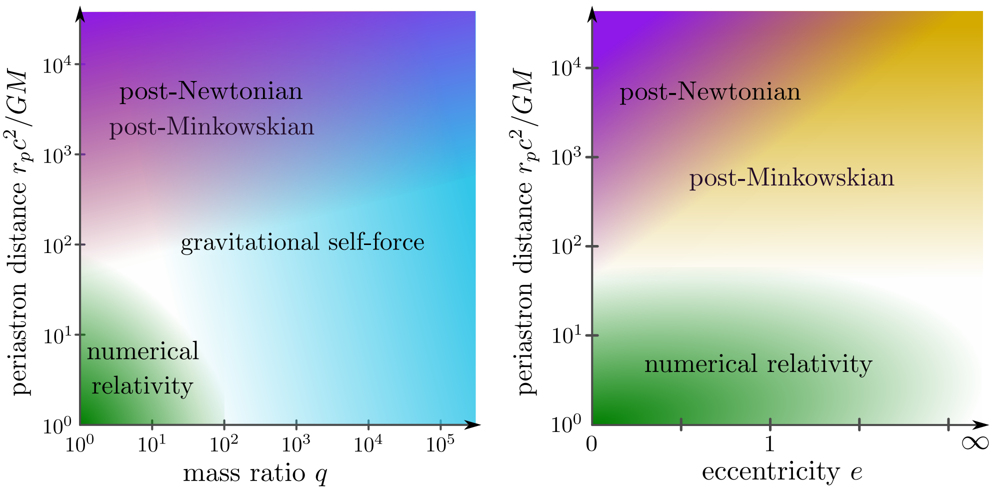


Figure 2. Regions of applicability of the PN, PM, and GSF approximations and NR in parameter space for **(left)** nearly circular orbits, and **(right)** comparable masses. Figure reproduced from Ref. [80].

The domains of applicability of the different approximation schemes (and numerical relativity) is illustrated in Fig. 2. The results from all three approximations are then combined using EOB formalism and the latter provides the entire GW signal (inspiral-merger-ringdown), which can be further calibrated using NR simulations.

We will follow the scattering-amplitudes approach to the relativistic two-body problem for unbounded orbits. Even though the idea of using scattering amplitudes to obtain classical results in gravity is not recent [81, 82], it might at first seem strange to start from a quantum theory. However, we only need to invoke the correspondence principle, which states that classical physics emerges from quantum theory in the limit of large quantum numbers. For the case of compact binary systems, the classical regime of quantum scattering amplitudes emerges in the limit of: (1) large angular momentum, $J \gg \hbar$, and (2) large gravitational charges $m_1, m_2 \gg M_{\text{Planck}}$ [83].

The amplitude framework presupposes that the separation between the compact objects is much larger than the typical size of the objects; thus, we can treat the two bodies as point particles. For black holes, the point-particle approximation seems appropriate given the black hole uniqueness theorem for asymptotically flat, stationary solutions of Einstein-Maxwell equations (see e.g. [84]). This theorem implies black holes are characterized by their mass, angular momentum, and electric charge – not unlike elementary particles. The dynamics of charged black holes (Reissner-Nordström) in the context of binary coalescences has received considerably less attention in the literature than the nonspinning (Schwarzschild) and spinning (Kerr) uncharged black hole solutions. This is because astrophysical black holes are expected to be electrically neutral within the Standard Model [85–87]. However, models of minicharged dark matter [88, 89] and dark photons [90, 91], could allow astrophysical black holes to be charged and be described formally by the Kerr-Newman black hole solution [92]. This makes the study of charged BBH systems not only a subject of formal interest, but also of relevance for future precision gravitational-wave detectors and motivates our study. Recent work in Einstein-Maxwell theory has seen the two-body potential computed at 2PN order [93], the three-body potential computed at 2PM order [94], and the scattering angle computed at 3PM order [95].

In the present work, we determine the two-body Hamiltonian for charged, spinless compact bodies at the 2PM order. The conservative dynamics is encoded in the four-point scattering amplitude of massive particles coupled to Einstein-Maxwell theory. From the Hamiltonian, we also compute the scattering angle of the two massive bodies at the 2PM order. Unless stated otherwise, we use units where $c = \hbar = \epsilon_0 = 1$, and use a mostly minus metric convention, $\mathrm{diag}(\eta_{\mu\nu}) = (1, -1, -1, -1)$.

2 Kinematics and QFT Framework

2.1 External Matter Kinematics

We describe the dynamics of two charged, spinless compact bodies by considering the scattering of two massive complex scalar fields, which interact via the exchange of massless photons and gravitons

$$\phi_1(-p_1, m_1) + \phi_2(-p_2, m_2) \to \phi_1(p_4, m_1) + \phi_2(p_3, m_2)\,. \tag{2.1}$$

We use an all-outgoing convention for external momenta, which is reflected above by the minus signs in the momenta of the initial states. The on-shell conditions for the external states are

$$p_1^2 = p_4^2 = m_1^2\,, \qquad p_2^2 = p_3^2 = m_2^2\,. \tag{2.2}$$

The scattering process is completely characterized in terms of the Mandelstam variables

$$s = (p_1 + p_2)^2\,, \quad t = (p_1 + p_4)^2\,, \quad u = (p_1 + p_3)^2\,, \tag{2.3}$$

subject to the constraint

$$s + t + u = 2m_1^2 + 2m_2^2\,. \tag{2.4}$$

Physical scattering processes correspond to the region $s > (m_1 + m_2)^2\,, t < 0\,,$ and $u < 0$. Given that only two Mandelstam variables are independent of each other, four-point scattering amplitudes are functions of only two invariants. We introduce the following Lorentz invariant quantity[1]

$$\sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}\,, \tag{2.5}$$

[1] This is simply the relativistic Lorentz factor characterizing the relative velocity between the two massive particles.

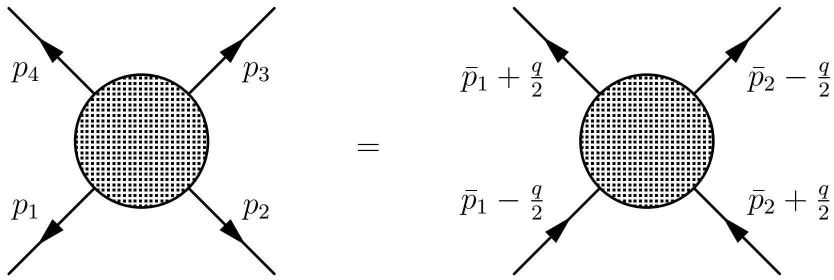


Figure 3. Parametrization of external momenta for scattering states.

and choose to parametrize the scattering amplitudes by the Lorentz invariants σ and $t = q^2$, where $q = p_1 + p_4$ is the four-momentum transfer. In terms of these two variables, the physical scattering region corresponds to $\sigma > 1\,, q^2 < 0\,, u < 0$.

We restrict our attention to the case of elastic scattering. For elastic scattering, the external momenta in the center of mass frame are given by

$$p_1 = \begin{pmatrix} -E_1 \\ -\boldsymbol{p} \end{pmatrix}, \quad p_2 = \begin{pmatrix} -E_2 \\ \boldsymbol{p} \end{pmatrix}, \quad p_3 = \begin{pmatrix} E_2 \\ -\boldsymbol{p}' \end{pmatrix}, \quad p_4 = \begin{pmatrix} E_1 \\ \boldsymbol{p}' \end{pmatrix}, \tag{2.6}$$

where $E_{1,2} = \sqrt{\boldsymbol{p}^2 + m_{1,2}^2}$, and Eq. (2.2) implies $\boldsymbol{p}^2 = \boldsymbol{p}'^2$. In the center of mass frame the momentum transfer is purely spatial, therefore $(-q^2) = \boldsymbol{q}^2$.

Following Ref. [96], we parametrize the external kinematics in terms of special Sudakov variables [97] in order to facilitate taking the classical limit of quantum scattering amplitudes later on. In terms of these new variables we have

$$p_1 = -\left(\bar{p}_1 - \frac{q}{2}\right), \quad p_2 = -\left(\bar{p}_2 + \frac{q}{2}\right), \quad p_3 = \bar{p}_2 - \frac{q}{2}, \quad p_4 = \bar{p}_1 + \frac{q}{2}. \tag{2.7}$$

The kinematics for the scattering process is summarized in Fig. 3.

It is easy to show that the barred momenta $\bar{p}_i$ are orthogonal to the momentum transfer q,

$$\bar{p}_1 \cdot q = \frac{1}{2}(p_4^2 - p_1^2) = 0\,, \quad \bar{p}_2 \cdot q = \frac{1}{2}(p_2^2 - p_3^2) = 0\,, \tag{2.8}$$

by virtue of Eq. (2.2). Given that $s = (\bar{p}_1 + \bar{p}_2)^2 = (p_1 + p_2)^2$, the physical scattering region is the same as before: $s > (m_1 + m_2)^2\,, t < 0\,, u < 0$. Additionally, we define barred masses

$$\bar{m}_1^2 = \bar{p}_1^2 = m_1^2 - \frac{q^2}{4}\,, \quad \bar{m}_2^2 = \bar{p}_2^2 = m_2^2 - \frac{q^2}{4}\,, \tag{2.9}$$

and four-velocities

$$u_1^\mu = \frac{\bar{p}_1^\mu}{\bar{m}_1}\,, \quad u_2^\mu = \frac{\bar{p}_2^\mu}{\bar{m}_2}\,. \tag{2.10}$$

Given the definition in Eq. (2.10), we deduce

$$u_1 \cdot q = u_2 \cdot q = 0\,, \quad u_1^2 = u_2^2 = 1\,, \tag{2.11}$$

from Eqs. (2.8) and (2.9), respectively. In analogy with Eq. (2.5), we introduce the dimensionless parameter

$$y = \frac{\bar{p}_1 \cdot \bar{p}_2}{\bar{m}_1 \bar{m}_2} = u_1 \cdot u_2 \,, \tag{2.12}$$

which is related to σ by the relation

$$y = \sigma + \frac{(m_1^2 + m_2^2)\sigma + 2m_1 m_2}{8m_1^2 m_2^2} q^2 + \mathcal{O}(q^4) \,. \tag{2.13}$$

In terms of this new variable, the physical scattering region in the s-channel is given by $y > 1 \,, q^2 < 0 \,, u < 0$.

2.2 Einstein-Maxwell Theory Coupled to Scalar Fields

A binary system of charged, spinless compact bodies with masses $m_{1,2}$ and electric charges $eQ_{1,2}$ interacting gravitationally and electromagnetically is described by two complex scalar fields $\phi_{1,2}$ coupled to Einstein-Maxwell theory. The action for the system is given by

$$S = S_{\text{EH}} + S_{\text{Maxwell}} + S_{\text{matter}} + S_{\text{GF}} \,. \tag{2.14}$$

The first term is the Einstein-Hilbert action

$$S_{\text{EH}} = -\frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} R \,, \tag{2.15}$$

where G is Newton's constant, R is the Ricci scalar, and g is the determinant of the metric $g_{\mu\nu}$. We will work in the weak-field regime and expand the metric around a flat Minkowski background $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \,, \qquad \kappa = \sqrt{32\pi G} \,, \tag{2.16}$$

and where the quantum fluctuation around this background is identified as the graviton field $h_{\mu\nu}$, a massless spin-2 field which mediates the gravitational interaction. From the expansion of the metric we obtain expansions for the inverse metric field $g^{\mu\nu}$ and for the determinant $\sqrt{-g}$, namely

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}{}_{\lambda} h^{\lambda\nu} + \cdots \,, \\ \sqrt{-g} &= 1 + \frac{\kappa}{2} h^{\mu}{}_{\mu} + \frac{\kappa^2}{8} \left((h^{\mu}{}_{\mu})^2 - 2h_{\mu\nu}h^{\mu\nu} \right) + \cdots \,. \end{aligned} \tag{2.17}$$

As a result of the weak-field expansion, an infinite number of terms are generated for the Einstein-Hilbert action which are organized as a series in powers of κ. Next, we have the Maxwell action

$$S_{\text{Maxwell}} = -\frac{1}{4} \int \mathrm{d}^4 x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \,, \tag{2.18}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\rho{}_{\mu\nu} A_\rho - \partial_\nu A_\mu + \Gamma^\rho{}_{\nu\mu} A_\rho$ is the field strength corresponding to the photon field A_μ. The connection components $\Gamma^\rho{}_{\mu\nu}$ are symmetric under interchange of μ and ν, which reduces the expression of the field strength to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Expanding the Maxwell action to first order in κ, we get

$$S^{(0)}_{\text{Maxwell}} = -\frac{1}{4} \int \mathrm{d}^4x \; F_{\mu\nu} F^{\mu\nu} \,, \tag{2.19}$$

$$S^{(1)}_{\text{Maxwell}} = -\frac{\kappa}{8} \int \mathrm{d}^4x \left(h^\lambda{}_\lambda F_{\mu\nu} F^{\mu\nu} + 4 h^{\mu\lambda} F_{\mu\nu} F^\nu{}_\lambda \right) \,. \tag{2.20}$$

Next, we have the matter sector of the action which reads

$$S_{\text{matter}} = \int \mathrm{d}^4x \sqrt{-g} \left[\sum_{i=1}^{2} g^{\mu\nu} (\nabla_\mu \phi_i + ieQ_i A_\mu \phi_i)^\dagger (\nabla_\nu \phi_i + ieQ_i A_\nu \phi_i) - m_i^2 \phi_i^\dagger \phi_i \right] \,, \tag{2.21}$$

where the electric charge Q_i is in multiples of the fundamental charge e, and for scalar fields we have $\nabla_\mu \phi_i = \partial_\mu \phi_i$. If we introduce the gauge covariant derivative

$$D_\mu \phi_i = (\partial_\mu + ieQ_i A_\mu) \phi_i \,, \tag{2.22}$$

then the first few terms in the matter action expansion are given by

$$S^{(0)}_{\text{matter}} = \int \mathrm{d}^4x \left[\sum_{i=1}^{2} (D_\mu \phi_i)^\dagger (D^\mu \phi_i) - m_i^2 \phi_i^\dagger \phi_i \right] \,, \tag{2.23}$$

$$S^{(1)}_{\text{matter}} = \frac{\kappa}{2} \int \mathrm{d}^4x \left[\sum_{i=1}^{2} h^\lambda{}_\lambda \left((D_\mu \phi_i)^\dagger (D^\mu \phi_i) - m_i^2 \phi_i^\dagger \phi_i \right) - 2 h^{\mu\nu} (D_\mu \phi_i)^\dagger (D_\nu \phi_i) \right] \,, \tag{2.24}$$

$$\begin{aligned} S^{(2)}_{\text{matter}} = \frac{\kappa^2}{8} \int \mathrm{d}^4x \Bigg[& \sum_{i=1}^{2} \left(h^\lambda{}_\lambda h^\sigma{}_\sigma - 2 h_{\rho\sigma} h^{\rho\sigma} \right) \left((D_\mu \phi_i)^\dagger (D^\mu \phi_i) - m_i^2 \phi_i^\dagger \phi_i \right) \\ & + 8 \left(h^{\mu\lambda} h^\nu{}_\lambda - \frac{1}{2} h^\lambda{}_\lambda h^{\mu\nu} \right) (D_\mu \phi_i)^\dagger (D_\nu \phi_i) \Bigg] \,. \end{aligned} \tag{2.25}$$

The last term in Eq. (2.14) is S_{GF} and denotes the gauge-fixing piece. This term is required in order to quantize the photon field A_μ, and the graviton field $h_{\mu\nu}$. Common gauge choices are the Lorentz gauge for the photon field and the *de Donder* gauge for the graviton field.

Since we will consider scattering amplitudes up to one-loop order, we only need expanded the Maxwell action to first order in κ and the matter action to second order in κ. The Feynman rules for the lowest order interactions vertices of photons and scalars with gravitons follow from Eqs. (2.20), (2.24), and (2.25).

We remark that the action in Eq. (2.14) is an effective description of the binary system in the point-particle approximation. This approximation is valid when the typical size of the objects R is much smaller than the separation r between them, i.e., $R \ll r$. Nonetheless, finite-size effects can be incorporated by including higher-dimension operators [98, 99].

2.3 Classical Limit of Scattering Amplitudes

One of the advantages of using quantum scattering amplitudes as our computational tool for the two-body problem in the PM expansion is that our results are manifestly relativistic, which makes them compact and valid to all orders in velocity. This approach, however, introduces the additional complication of having to identify and extract the classical physics encoded in scattering amplitudes.

From the correspondence principle, we know that the classical limit is the limit of large quantum numbers. In the case of a binary system of compact objects, this is the regime of large angular momentum $J \gg 1$ (in natural units) and of large gravitational masses $m_1, m_2 \gg M_{\text{Planck}}$. The limit of large angular momentum is equivalent to the limit in which the particles' minimal separation[2] $|\boldsymbol{b}|$ is much larger than the de Broglie wavelength, λ_{dB}, of each particle,

$$|\boldsymbol{b}| \gg \lambda_{\text{dB}} = \frac{1}{|\boldsymbol{p}|} \iff J \sim |\boldsymbol{p} \times \boldsymbol{b}| \gg 1 \,. \tag{2.26}$$

Adding the fact that $|\boldsymbol{b}|$ is the Fourier conjugate to the momentum transfer $\boldsymbol{q}$, we have the following kinematic hierarchy of scales

$$m \sim \sqrt{s}, m_1, m_2, |\boldsymbol{p}| \sim J|\boldsymbol{q}| \gg |\boldsymbol{q}| \,, \tag{2.27}$$

where m stands for some characteristic hard scale in the scattering process. From the above hierarchy, we can construct the dimensionless parameter $\lambda = |\boldsymbol{q}|/m \ll 1$, which is the expansion parameter that controls classical versus quantum corrections. This last statement comes from the fact that restoring factors of $\hbar$ shows that $\lambda \sim \hbar$, and so additional powers in λ correspond to higher quantum corrections. However, instead of using λ as our small parameter to count powers of $\hbar$ we will use $|\boldsymbol{q}|$ and talk of the small-$|\boldsymbol{q}|$ expansion of scattering amplitudes.

In the small-$|\boldsymbol{q}|$ expansion, classical, long-distance physics manifests itself in scattering amplitudes as non-analytic terms in $|\boldsymbol{q}|$, e.g., $1/|\boldsymbol{q}|^2, 1/|\boldsymbol{q}|, \ln|\boldsymbol{q}|$, etc. This is because, after Fourier transform, these terms give non-local effects in the form of $1/r^n$ potentials, as opposed to analytic terms in $|\boldsymbol{q}|$ which give local effects in the form of $\delta(r)$-functions and correspond to short-distance interactions. Thus, we can disregard any analytic terms and only focus on the non-analytic ones.

The classical contributions in scattering amplitudes can be isolated and extracted by using the method of regions [100, 101]. In the context of the method of regions, we learn that only the *potential region* yields the classical conservative dynamics we

[2] For a scattering process the relevant scale for the minimal separation is the impact parameter.

are after [59, 63, 102]. In the potential region, the four-momentum components of internal photon and graviton lines scale as $\ell = (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}|)$, where v is some small velocity parameter, $|\boldsymbol{v}| \ll 1$. By using this scaling for loop momenta, we can expand loop integrands we encounter in scattering amplitudes in the potential region. Following Ref. [96], this is accomplished as follows. First, we expand loop integrands in the *soft* region; that is, we perform a small-$|q|$ expansion of the integrand. This procedure is facilitated by using the special kinematic parametrization introduced in Eq. (2.7) and leads to the hierarchy of scales

$$m \sim \sqrt{s}, m_1, m_2, |\bar{p}_i| \gg |q| \sim |\ell| \,. \tag{2.28}$$

In the soft region, photon and graviton propagators have a trivial expansion since they both have homogeneous scaling in powers of $|q|$, but matter propagators do not have a homogeneous $|q|$-scaling and have to be expanded,

$$\frac{1}{(\ell - p_i)^2 - m_i^2} = \frac{1}{\ell^2 - 2p_i \cdot \ell} = \frac{1}{\ell^2 + 2\bar{p}_i \cdot \ell \pm \ell \cdot q} = \frac{1}{\bar{m}_i}\frac{1}{2u_i \cdot \ell} - \frac{1}{\bar{m}_i^2}\frac{\ell^2 \pm \ell \cdot q}{(2u_i \cdot \ell)^2} + \cdots \,, \tag{2.29}$$

so that each term in the expansion is homogenous in $|q| \sim |\ell|$. After the soft expansion, we expand in the potential region – a subregion of the soft region. In the potential region, the normalized velocity four-vectors u_i scale as $u_i^\mu = (u_i^0, \boldsymbol{u}_i) \sim (1, |\boldsymbol{v}|)$. The expansion of the massless and matter propagators is given by

$$\frac{1}{\ell^2} = \frac{1}{\omega^2 - \boldsymbol{\ell}^2} = -\frac{1}{\boldsymbol{\ell}^2} - \frac{\omega^2}{(\boldsymbol{\ell}^2)^2} - \frac{\omega^4}{(\boldsymbol{\ell}^2)^3} - \cdots \,, \tag{2.30}$$

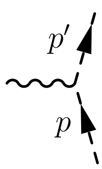
$$\frac{1}{2u_i \cdot \ell} = \frac{1}{2(u_i^0 \omega - \boldsymbol{u}_i \cdot \boldsymbol{\ell})} \,. \tag{2.31}$$

The resulting integrals can then be evaluated by traditional integration methods, by using computer software, or by the method of differential equations [75–77]. Integrands only have to be expanded to leading order in the velocity, $|\boldsymbol{v}|$.

3 Scattering Amplitudes in the Potential Region

3.1 Tree-level Amplitudes

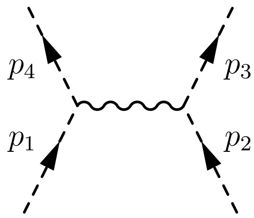
We start by computing the $2 \to 2$ tree-level scattering amplitude between two charged scalars in scalar Quantum Electrodynamics (SQED). Recall that we will work in an all-outgoing convention for external momenta. The relevant Feynman vertex rule is the 2-scalar-1-photon vertex given by



$$= -ieQ_i(p' - p)^\mu \,. \tag{3.1}$$

In turn, the scalar QED tree-level amplitude is given by

$$\mathcal{M}^{\text{tree}}_{e^2}(\sigma, q^2) =$$



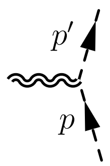
$$= \frac{e^2 Q_1 Q_2}{q^2}(4p_1 \cdot p_2 + q^2) \tag{3.2}$$

$$= \frac{e^2 Q_1 Q_2}{q^2}(4m_1 m_2 \sigma + q^2)\,,$$

where in the last equation we used Eq. (2.5). Going back to our discussion of Sec. 2.3, we disregard any terms in the amplitude which are analytic in q^2. Hence, the scalar QED tree-level amplitude in the conservative (potential) region, which we denote with a subscript (p), is then given by

$$\mathcal{M}^{\text{tree}}_{e^2,(p)}(\sigma, q^2) = \frac{e^2 Q_1 Q_2}{q^2}(4m_1 m_2 \sigma)\,. \tag{3.3}$$

Next, we compute the tree-level scattering amplitude in gravity due to a single graviton exchange. For this, we need the 2-scalar-1-graviton vertex rule



$$= \frac{i\kappa}{2}\left[p^\mu p'^\nu + p^\nu p'^\mu - \eta^{\mu\nu}(p \cdot p' + m_i^2)\right]\,, \tag{3.4}$$

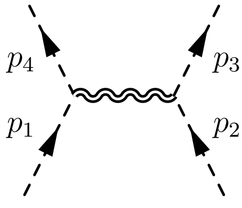
as well as the graviton propagator which, in $D = 4$ space-time dimensions and in the de Donder gauge, takes the form [103, 104]

$$\mu\nu$$

$$\alpha\beta \quad = \frac{iP^{\mu\nu\alpha\beta}}{q^2 + i\varepsilon} = \frac{i}{q^2 + i\varepsilon}\frac{1}{2}\left[\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right]\,. \tag{3.5}$$

The tree-level scattering amplitude is then

$$\mathcal{M}^{\text{tree}}_{G}(\sigma, q^2) =$$



$$= \frac{16\pi G}{q^2}\left[m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2 - q^2(p_1 \cdot p_2)\right] \tag{3.6}$$

$$= \frac{16\pi G m_1^2 m_2^2}{q^2}\left(1 - 2\sigma^2 - \frac{\sigma q^2}{m_1 m_2}\right)\,,$$

and so the corresponding amplitude in the potential region is

$$\mathcal{M}^{\text{tree}}_{G,(p)}(\sigma, q^2) = \frac{16\pi G m_1^2 m_2^2}{q^2}(1-2\sigma^2)\,. \tag{3.7}$$

Combining Eqs. (3.3) and (3.7), the scattering amplitude in Einstein-Maxwell theory at 1PM order in the potential region is given by

$$\mathcal{M}_1 = \frac{16\pi G m_1^2 m_2^2}{q^2}(1-2\sigma^2) + \frac{e^2 Q_1 Q_2}{q^2}(4m_1 m_2 \sigma)\,. \tag{3.8}$$

3.1.1 Restoring $\hbar$

Before moving on to one-loop amplitudes, let us examine the $\hbar$-scaling of the tree-level amplitudes we just calculated. This will determine the order in $\hbar$ which produces classical results since the Fourier transform of Eqs. (3.3) and (3.7) gives rise to the nonrelativistic Coulomb and Newtonian potentials, respectively.

When $\hbar$ is restored, we must look at the dimensions of the coupling constants. In electrodynamics and gravity, a factor of $1/\sqrt{\hbar}$ multiplies the couplings: $e/\sqrt{\hbar} = \sqrt{4\pi\alpha}$, $\kappa = \sqrt{32\pi G/\hbar}$. In addition, we learned in Sec. 2.3 that counting powers of $\hbar$ is equivalent to counting powers in $|q|$. Putting it all together, we find that by making factors of $\hbar$ explicit

$$\begin{aligned}
\mathcal{M}^{\text{tree}}_{e^2,(p)}(\sigma, q^2) &= \frac{e^2 Q_1 Q_2}{\hbar^3 q^2}(4m_1 m_2 \sigma)\,,\\
\mathcal{M}^{\text{tree}}_{G,(p)}(\sigma, q^2) &= \frac{16\pi G m_1^2 m_2^2}{\hbar^3 q^2}(1-2\sigma^2)\,.
\end{aligned} \tag{3.9}$$

Hence, classical contributions in scattering amplitudes scale as $\hbar^{-3}$. Quantum corrections, then, correspond to terms that scale with additional powers in $\hbar$.

3.2 Feynman Integrals

3.2.1 Notation and Conventions of Feynman Integrals

A D-dimensional L-loop Feynman integral is an integral of the form [105]

$$I(p_1,\ldots,p_E; m_1^2,\ldots,m_n^2; \underline{\nu}; D) = \int \left(\prod_{j=1}^{L} \mu^{2\epsilon}\frac{\mathrm{d}^D\ell_j}{(2\pi)^D}\right) \frac{\mathcal{N}(\{\ell_j\cdot\ell_k, \ell_j\cdot p_k\})}{\prod_{j=1}^{n}(q_j^2 - m_j^2 + i\varepsilon)^{\nu_j}}\,. \tag{3.10}$$

The integral is a function of the external momenta $p_1,\ldots,p_E$ and propagator masses $m_1^2,\ldots,m_n^2$, where $\underline{\nu} = (\nu_1,\ldots,\nu_n)$ is a vector of integers and D is the space-time dimension, which we take to be $D = 4-2\epsilon$. Evaluating Feynman integrals in general D dimensions is a *regularization scheme* called *dimensional regularization* [106, 107]. Its purpose is to regulate ultraviolet (UV) or infrared (IR) divergences that one encounters when computing loop integrals. The measure is multiplied by $\mu^{2\epsilon}$, which is a dimensionful scale introduced so that $\mu^{2\epsilon}\mathrm{d}^D\ell_j$ in $D = 4 - 2\epsilon$ dimensions has the same mass dimension as the measure $\mathrm{d}^4\ell_j$ in physical $D = 4$ dimensions. The numerator factor $\mathcal{N}$ is a Lorentz invariant polynomial of dot products between internal loop momenta, or between internal loop momenta and an external momenta. Finally, the momenta q_j appearing in the propagators are linear combinations of the external momenta and the internal loop momenta

$$q_j = \sum_{i=1}^{L} \alpha_{ji}\ell_i + \sum_{i=1}^{E-1} \beta_{ji}p_i\,, \qquad \alpha_{ji}, \beta_{ji} \in \{-1, 0, +1\}\,, \tag{3.11}$$

where the second sum goes to $E-1$ due to momentum conservation $\sum_{i=1}^{E} p_i = 0$.

We will adopt a different normalization for the integration measure factor than that of Eq. (3.10). Following the convention of Ref. [108], we remove a factor of $i(4\pi e^{-\gamma_E})^\epsilon/(4\pi)^2$ per loop, and define

$$(\bar{\mu}^2)^\epsilon \equiv (4\pi e^{-\gamma_E}\mu^2)^\epsilon\,, \tag{3.12}$$

where γ_E is the Euler-Mascheroni constant, so that

$$\mu^{2\epsilon}\frac{\mathrm{d}^D\ell_j}{(2\pi)^D} = \frac{i}{(4\pi)^2}\bar{\mu}^{2\epsilon}e^{\gamma_E\epsilon}\frac{\mathrm{d}^D\ell_j}{i\pi^{D/2}}\,. \tag{3.13}$$

To summarize, we will be evaluating Feynman integrals of the form

$$I(p_1,\ldots,p_E; m_1^2,\ldots,m_n^2; \underline{\nu}; D) = \int\left(\prod_{j=1}^{L}\frac{\mathrm{d}^D\ell_j e^{\gamma_E\epsilon}}{i\pi^{D/2}}\right)\frac{\mathcal{N}(\{\ell_j\cdot\ell_k, \ell_j\cdot p_k\})}{\prod_{j=1}^{n}(q_j^2 - m_j^2 + i\varepsilon)^{\nu_j}}\,, \tag{3.14}$$

and multiply the end result by a factor of $i\bar{\mu}^{2\epsilon}/(4\pi)^2$ per loop to recover the physical integration measure factor $1/(2\pi)^D$.

3.2.2 Power counting in $\hbar$ of one-loop integrals

Recall that classical contributions from amplitudes scale as $\hbar^{-3}$. Given the scaling of the electromagnetic and gravitational couplings $1/\sqrt{\hbar}$ and the fact that one-loop diagrams carry four powers of the coupling, this means we have a factor $\hbar^{-2}$ alone just from the couplings. The remaining factor of $\hbar^{-1}$ required of classical pieces can therefore only arise from non-analytic terms in the amplitude of the form $1/\sqrt{-q^2}$.

At the one-loop order, the square root non-analytic term arises from diagrams with two photon propagators in scalar QED and two graviton propagators in gravity [109], which reduces the topology[3] of Feynman integrals that have to be considered to just two: the box and crossed box topologies. However, when using the barred momenta of Eq. (2.7) the crossed box topology is related to the box topology through

[3] An integral topology is just a list of the linearly independent propagators present in a given family of Feynman integrals.

the mapping $u_1 \to u_1, u_2 \to -u_2$. Therefore, we only really have to consider the box topology.

A generic integral in the box topology is of the form

$$I^{\mathrm{II}}_{\underline{\nu}} = \int \frac{\mathrm{d}^D\ell e^{\gamma_E\epsilon}}{i\pi^{D/2}} \frac{1}{[(\ell-p_1)^2-m_1^2]^{\nu_1}[(\ell+p_2)^2-m_2^2]^{\nu_2}[\ell^2]^{\nu_3}[(\ell-q)^2]^{\nu_4}}\,, \tag{3.15}$$

where we have omitted the full functional dependence of the Feynman integrals and left the Feynman $i\varepsilon$-prescription implicit for the sake of notational simplicity. In addition, we will indicate the vector of propagator powers as a subscript.

If one of the propagator indices is a non-positive integer, the box integral reduces to a triangle integral; if two of the propagator indices are non-positive, we obtain a bubble integral; if three propagator indices are non-positive, we are left with a tadpole integral. Since we need two massless propagators to generate the square root non-analytic term, we can disregard the tadpole integral.

We can also ignore the bubble integral because of its $\hbar$-scaling. Recall that massless propagators remain unchanged when expanding in the soft region. If we now restore factors of $\hbar$ and use the fact that the loop momentum ℓ scales like the momentum transfer q in the soft region given the hierarchy of scales in Eq. (2.28), we find[4]

$$i\mathcal{M}^{1-\mathrm{loop}}_{\mathrm{bubble}} \sim \frac{g^2}{\hbar^2} I^{\mathrm{II}}_{0,0,1,1} \sim \frac{g^2}{\hbar^2}\int \hbar^4 \mathrm{d}^D\ell \frac{1}{\hbar^2\ell^2}\frac{1}{\hbar^2(\ell-q)^2}\,, \tag{3.16}$$

where g is the coupling constant, either α or G. This shows that the bubble integral scales as $\hbar^{-2}$, so it does not produce any classical contribution. A similar analysis shows that the triangle and box integrals have appropriate scalings to contain classical pieces, as we will see explicitly in the following subsection.

It will be useful for later to introduce the integral family with linearized matter propagators

$$J^{\mathrm{II}}_{\underline{\nu}} = \int \frac{\mathrm{d}^D\ell e^{\gamma_E\epsilon}}{i\pi^{D/2}} \frac{1}{[2u_1\cdot\ell]^{\nu_1}[-2u_2\cdot\ell]^{\nu_2}[\ell^2]^{\nu_3}[(\ell-q)^2]^{\nu_4}}\,. \tag{3.17}$$

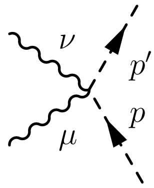
3.3 Scalar QED One-loop Amplitudes

Insofar as classical contributions are concerned, the one-loop amplitude can be decomposed as

$$\mathcal{M}^{1-\mathrm{loop}}_{e^4}(\sigma,q^2) = \mathcal{M}^{\triangleright}_{e^4} + \mathcal{M}^{\triangleleft}_{e^4} + \mathcal{M}^{\square}_{e^4} + \mathcal{M}^{\boxtimes}_{e^4}\,, \tag{3.18}$$

[4] We scale the integration measure in powers of $\hbar$ in dimensional regularization as if we were working in $D=4$ dimensions.

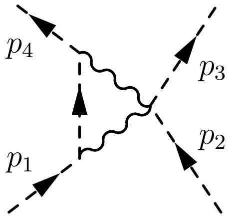
that is, a sum of two triangle diagrams, a box diagram, and a crossed box diagram. For the triangle diagrams, we will need the 2-scalar-2-photon vertex rule given by



$$= 2ie^2Q_i^2\eta^{\mu\nu}\,. \tag{3.19}$$

The right-triangle diagram

$$i\mathcal{M}^{\triangleright}_{e^4}(\sigma, q^2) =$$



$$, \tag{3.20}$$

has an associated Feynman integral given by

$$i\mathcal{M}^{\triangleright}_{e^4} = 2e^4Q_1^2Q_2^2\int\frac{\mathrm{d}^D\ell e^{\gamma_E\epsilon}}{i\pi^{D/2}}\frac{(2p_1-\ell)\cdot(-2p_1+q+\ell)}{[(\ell-p_1)^2-m_1^2+i\varepsilon]\ell^2(\ell-q)^2}\,. \tag{3.21}$$

To extract the classical contributions, we follow our discussion of Sec. 2.3. First, we expand the integrand in the soft region. Using Eq. (2.7), we can rewrite the integrand in terms of the barred momenta, simplify the numerator product, and obtain the integral

$$i\mathcal{M}^{\triangleright}_{e^4} = 2e^4Q_1^2Q_2^2\int\frac{\mathrm{d}^D\ell e^{\gamma_E\epsilon}}{i\pi^{D/2}}\frac{-4\bar{m}_1^2-4\bar{p}_1\cdot\ell+\ell\cdot q-\ell^2}{[\ell^2+2\bar{p}_1\cdot\ell-\ell\cdot q+i\varepsilon]\ell^2(\ell-q)^2}\,. \tag{3.22}$$

The next step is to expand the massive propagator in the soft region using Eq. (2.29) so the resulting terms have homogeneous $|q|$-scaling

$$\begin{aligned}i\mathcal{M}^{\triangleright}_{e^4} = 2e^4Q_1^2Q_2^2\int\frac{\mathrm{d}^D\ell e^{\gamma_E\epsilon}}{i\pi^{D/2}}\frac{-4\bar{m}_1^2-4\bar{p}_1\cdot\ell+\ell\cdot q-\ell^2}{\ell^2(\ell-q)^2}\Bigg[&\frac{1}{\bar{m}_1}\frac{1}{2u_1\cdot\ell+i\varepsilon}\\ &-\frac{1}{\bar{m}_1^2}\frac{\ell^2-\ell\cdot q}{(2u_1\cdot\ell+i\varepsilon)^2}+\cdots\Bigg]\,.\end{aligned} \tag{3.23}$$

By restoring factors of $\hbar$, we determine that the leading order term in $\hbar$ scales classically, i.e. as $\hbar^{-3}$, and is given by

$$i\mathcal{M}^{\triangleright}_{e^4} = 2e^4Q_1^2Q_2^2\int\frac{\mathrm{d}^D\ell e^{\gamma_E\epsilon}}{i\pi^{D/2}}\frac{-4\bar{m}_1}{(2u_1\cdot\ell+i\varepsilon)\ell^2(\ell-q)^2} = (-8\bar{m}_1e^4Q_1^2Q_2^2)J^{\mathrm{II}}_{1,0,1,1}\,. \tag{3.24}$$

Having identified which term contributes to the classical physics, we then expand the integrand in the potential region by using Eqs. (2.30) and (2.31). For convenience, we will perform the calculation in the following reference frame

$$u_1 = (1,0,0,0)\,, \quad u_2 = (\sqrt{1+v^2},0,0,v)\,, \quad q = (0,\boldsymbol{q},0)\,, \tag{3.25}$$

where we had to careful so Eq. (2.11) is satisfied, and where $v = \sqrt{y^2-1}$. Our integral expanded to leading order in the potential region in this frame takes the form

$$J^{\mathrm{II},(p)}_{1,0,1,1} = \frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \int \frac{\mathrm{d}^{D-1}\boldsymbol{\ell}}{[-\boldsymbol{\ell}^2][-(\boldsymbol{\ell}-\boldsymbol{q})^2]} \times \int \frac{\mathrm{d}\omega}{2\omega + i\varepsilon}\,. \tag{3.26}$$

The ω integral can be performed by symmetrizing over the energy component

$$\int \frac{\mathrm{d}\omega}{2\omega+i\varepsilon} \to \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\omega+i\varepsilon} + \frac{1}{2}\int \frac{\mathrm{d}\omega}{-2\omega+i\varepsilon} = \frac{-i}{2}\int \mathrm{d}\omega \frac{\varepsilon/2}{\omega^2+(\varepsilon/2)^2} = \frac{-i\pi}{2}\,. \tag{3.27}$$

We are left with only an integral over the spatial components, which can be written in the form

$$J^{\mathrm{II},(p)}_{1,0,1,1} = -\frac{e^{\gamma_E \epsilon}\sqrt{\pi}}{2} \int \frac{\mathrm{d}^{D-1}\boldsymbol{\ell}}{\pi^{(D-1)/2}} \frac{1}{\boldsymbol{\ell}^2(\boldsymbol{\ell}-\boldsymbol{q})^2}\,. \tag{3.28}$$

This integral is given in Ref. [108] and the final result reads

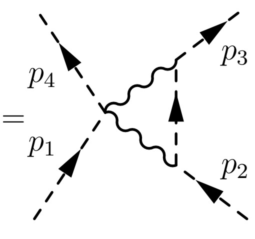
$$J^{\mathrm{II},(p)}_{1,0,1,1} = -(-q^2)^{-\frac{1}{2}-\epsilon} e^{\gamma_E \epsilon} \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)^2\Gamma(\frac{1}{2}+\epsilon)}{2\Gamma(1-2\epsilon)}\,. \tag{3.29}$$

Multiplying this result with the numerator factor of Eq. (3.24) and by the factor $i\bar{\mu}^{2\epsilon}/(4\pi)^2$, we conclude that the right-triangle diagram in the potential region is given by

$$\mathcal{M}^{\triangleright}_{e^4,(p)} = 8e^4 Q_1^2 Q_2^2 \frac{1}{(4\pi)^2} \left(\frac{-q^2}{\bar{\mu}^2}\right)^{-\epsilon} \frac{\sqrt{\pi}\bar{m}_1}{\sqrt{-q^2}} \frac{e^{\gamma_E \epsilon}\Gamma(\frac{1}{2}-\epsilon)^2\Gamma(\frac{1}{2}+\epsilon)}{2\Gamma(1-2\epsilon)}\,. \tag{3.30}$$

The same procedure when applied to the left-triangle diagram

$$i\mathcal{M}^{\triangleleft}_{e^4}(\sigma, q^2) = \tag{3.31}$$



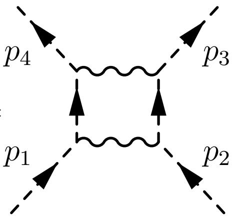
,

reveals an equivalent expression to the one we obtained for the right-triangle diagram but exchanging $\bar{m}_1$ and $\bar{m}_2$

$$\mathcal{M}^{\triangleleft}_{e^4,(p)} = 8e^4 Q_1^2 Q_2^2 \frac{1}{(4\pi)^2} \left(\frac{-q^2}{\bar{\mu}^2}\right)^{-\epsilon} \frac{\sqrt{\pi}\bar{m}_2}{\sqrt{-q^2}} \frac{e^{\gamma_E \epsilon}\Gamma(\frac{1}{2}-\epsilon)^2\Gamma(\frac{1}{2}+\epsilon)}{2\Gamma(1-2\epsilon)}\,. \tag{3.32}$$

The box diagram

$$i\mathcal{M}^{\square}_{e^4}(\sigma, q^2) = \quad , \tag{3.33}$$



has an associated Feynman integral given by

$$i\mathcal{M}^{\square}_{e^4} = e^4 Q_1^2 Q_2^2 \int \frac{\mathrm{d}^D \ell e^{\gamma_E \epsilon}}{i\pi^{D/2}} \frac{\Big[(2p_1 - \ell)\cdot(2p_2 + \ell)\Big]\Big[(-2p_1 + q + \ell)\cdot(-2p_2 - q - \ell)\Big]}{[(\ell - p_1)^2 - m_1^2 + i\varepsilon][(\ell + p_2)^2 - m_2^2 + i\varepsilon]\ell^2(\ell - q)^2} . \tag{3.34}$$

Expanding in the soft region, we obtain

$$\begin{aligned} i\mathcal{M}^{\square}_{e^4} = e^4 Q_1^2 Q_2^2 \int \frac{\mathrm{d}^D \ell e^{\gamma_E \epsilon}}{i\pi^{D/2}} & \frac{[4\bar{p}_1\cdot\bar{p}_2 + 2(\bar{p}_2 - \bar{p}_1)\cdot\ell - \ell^2][4\bar{p}_1\cdot\bar{p}_2 + 2(\bar{p}_2 - \bar{p}_1)\cdot\ell - (\ell - q)^2]}{\ell^2(\ell - q)^2} \\ & \times \left(\frac{1}{\bar{m}_1}\frac{1}{2u_1\cdot\ell + i\varepsilon} - \frac{1}{\bar{m}_1^2}\frac{\ell^2 - \ell\cdot q}{(2u_1\cdot\ell + i\varepsilon)^2} + \cdots \right) \\ & \times \left(\frac{1}{\bar{m}_2}\frac{1}{-2u_2\cdot\ell + i\varepsilon} - \frac{1}{\bar{m}_2^2}\frac{\ell^2 - \ell\cdot q}{(-2u_2\cdot\ell + i\varepsilon)^2} + \cdots \right) . \end{aligned} \tag{3.35}$$

If we now restore factors of $\hbar$, the leading order term in $\hbar$ scales not as $\hbar^{-3}$, but as $\hbar^{-4}$. This type of terms that exhibit an enhanced classical scaling are referred to as *superclassical* terms. Superclassical terms cancel when computing physical observables, so they do not have any physical significance. Even so, we still need to compute them since they are required to cancel in the matching procedure between the full theory and the effective field theory that we will discuss in Sec. 4.

The box superclassical piece in Eq. (3.35) is

$$\begin{aligned} 16e^4 Q_1^2 Q_2^2 \frac{(\bar{p}_1\cdot\bar{p}_2)^2}{\bar{m}_1\bar{m}_2} \int \frac{\mathrm{d}^D \ell e^{\gamma_E \epsilon}}{i\pi^{D/2}} & \frac{1}{(2u_1\cdot\ell + i\varepsilon)(-2u_2\cdot\ell + i\varepsilon)\ell^2(\ell - q)^2} \\ & = 16e^4 Q_1^2 Q_2^2 \bar{m}_1\bar{m}_2 y^2 J^{\mathrm{II}}_{1,1,1,1} \end{aligned} \tag{3.36}$$

Expanding the integral in the potential region at leading order in v in the frame of Eq. (3.25), we get

$$J^{\mathrm{II},(p)}_{1,1,1,1} = \int \frac{e^{\gamma_E \epsilon}\mathrm{d}^{D-1}\boldsymbol{\ell}}{i\pi^{D/2}} \frac{1}{\boldsymbol{\ell}^2(\boldsymbol{\ell} - \boldsymbol{q})^2} \int \frac{\mathrm{d}\omega}{(2\omega + i\varepsilon)(-2\omega + 2v\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell} + i\varepsilon)} , \tag{3.37}$$

where $\hat{\boldsymbol{z}}$ is the unit vector in the z-direction. The ω integral is performed by closing the contour in the complex ω-plane in, say, the upper half plane and using the residue theorem. We obtain

$$J^{\mathrm{II},(p)}_{1,1,1,1} = -\frac{e^{\gamma_E \epsilon}\sqrt{\pi}}{v} \int \frac{\mathrm{d}^{D-1}\boldsymbol{\ell}}{\pi^{(D-1)/2}} \frac{1}{(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell} + i\varepsilon)\boldsymbol{\ell}^2(\boldsymbol{\ell} - \boldsymbol{q})^2} . \tag{3.38}$$

The spatial integral can be found in Ref. [108], and the result is

$$J^{\mathrm{II},(p)}_{1,1,1,1} = \frac{1}{\sqrt{y^2-1}}(-q^2)^{-1-\epsilon}e^{\gamma_E\epsilon}\frac{i\pi\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{2\Gamma(-2\epsilon)}\,, \tag{3.39}$$

where we substituted $v=\sqrt{y^2-1}$. Multiplying this expression by its numerator factor $16e^4Q_1^2Q_2^2\bar{m}_1\bar{m}_2y^2$ and by $i\bar{\mu}^{2\epsilon}/(4\pi)^2$, we conclude that the box superclassical term is given by

$$\mathcal{M}^{\square(-4)}_{e^4,(p)} = 16e^4Q_1^2Q_2^2\frac{1}{(4\pi)^2}\left(\frac{-q^2}{\bar{\mu}^2}\right)^{-\epsilon}\frac{1}{(-q^2)}\frac{i\pi\bar{m}_1\bar{m}_2y^2}{\sqrt{y^2-1}}\frac{e^{\gamma_E\epsilon}\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{2\Gamma(-2\epsilon)}\,, \tag{3.40}$$

where the superscript (-4) indicates its $\hbar$-scaling.

Going back to Eq. (3.35), the classical terms are the next-to-leading order terms in $\hbar$ in the expression. After a little algebra, these terms are given by

$$\begin{aligned} e^4Q_1^2Q_2^2\int\frac{\mathrm{d}^D\ell e^{\gamma_E\epsilon}}{i\pi^{D/2}}\Bigg[&-\frac{8\bar{m}_2y}{(2u_1\cdot\ell+i\varepsilon)\ell^2(\ell-q)^2}-\frac{8\bar{m}_1y}{(-2u_2\cdot\ell+i\varepsilon)\ell^2(\ell-q)^2}\\ &+\frac{8\bar{m}_2y^2q^2}{(2u_1\cdot\ell+i\varepsilon)^2(-2u_2\cdot\ell+i\varepsilon)\ell^2(\ell-q)^2}+\frac{8\bar{m}_1y^2q^2}{(2u_1\cdot\ell+i\varepsilon)(-2u_2\cdot\ell+i\varepsilon)^2\ell^2(\ell-q)^2}\\ &-\frac{8\bar{m}_2y^2}{(2u_1\cdot\ell+i\varepsilon)^2(-2u_2\cdot\ell+i\varepsilon)\ell^2}-\frac{8\bar{m}_1y^2}{(2u_1\cdot\ell+i\varepsilon)(-2u_2\cdot\ell+i\varepsilon)^2\ell^2}\\ &-\frac{8\bar{m}_2y^2}{(2u_1\cdot\ell+i\varepsilon)^2(-2u_2\cdot\ell+i\varepsilon)(\ell-q)^2}-\frac{8\bar{m}_1y^2}{(2u_1\cdot\ell+i\varepsilon)(-2u_2\cdot\ell+i\varepsilon)^2(\ell-q)^2}\Bigg]. \end{aligned} \tag{3.41}$$

Above, we have highlighted in red and blue the terms whose integrals are zero. The integrals are zero because they are *scaleless*, and scaleless integrals vanish in dimensional regularization [105]. A scaleless integral is an integral that does not depend on any external scale, like masses or dot products of external momenta. By virtue of Eq. (2.11), the only dimensionful scale that remains after the soft expansion is q^2. The terms in red plainly do not depend on q^2 and so are scaleless; therefore, they vanish. For the terms in blue, by doing a change of variables $\ell^\mu\to\ell^\mu+q^\mu$ and using the orthogonality property in Eq. (2.11), we can show they are identical to the terms in red and so they vanish too.

We are left with the expression

$$e^4Q_1^2Q_2^2\Big[-8y\left(\bar{m}_2J^{\mathrm{II}}_{1,0,1,1}+\bar{m}_1J^{\mathrm{II}}_{0,1,1,1}\right)+8y^2q^2\left(\bar{m}_2J^{\mathrm{II}}_{2,1,1,1}+\bar{m}_1J^{\mathrm{II}}_{1,2,1,1}\right)\Big]\,, \tag{3.42}$$

where we now have to expand and evaluate the integrals in the potential region. The linearized triangle integral $J^{\mathrm{II},(p)}_{1,0,1,1}$ is given in Eq. (3.29), and the other linearized triangle integral $J^{\mathrm{II},(p)}_{0,1,1,1}$ is identical to the former

$$J^{\mathrm{II},(p)}_{0,1,1,1} = J^{\mathrm{II},(p)}_{1,0,1,1}\,, \tag{3.43}$$

since we get identical spatial integrals after performing the energy integrals. The remaining unknown integrals can be expressed in terms of simpler known integrals by using integration-by-parts (IBP) relations [74]. These relations are a consequence of the fact that surface terms vanish in dimensional regularization. To be concrete, given Eq. (3.10) for a generic D-dimensional L-loop Feynman integral, then

$$\int \mathrm{d}^D \ell_i \frac{\partial}{\partial \ell_i^\mu} \left[v^\mu \frac{\mathcal{N}(\{\ell_j \cdot \ell_k, \ell_j \cdot p_k\})}{\prod_{j=1}^n (q_j^2 - m_j^2 + i\varepsilon)^{\nu_j}} \right] = 0 \,, \qquad 1 \le i \le L \,, \tag{3.44}$$

where v^μ is any Lorentz vector, e.g., external momenta or internal loop momenta. The above type of equations will give rise to linear relations in the propagator exponents between Feynman integrals of a given family.

Using IBP relations it can be shown that

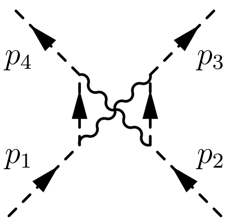
$$J^{\mathrm{II},(p)}_{2,1,1,1} = J^{\mathrm{II},(p)}_{1,2,1,1} = \frac{4-D}{(y-1)q^2} J^{\mathrm{II},(p)}_{1,0,1,1} = \frac{2\epsilon}{(y-1)q^2} J^{\mathrm{II},(p)}_{1,0,1,1} \,. \tag{3.45}$$

Putting everything together in Eq. (3.42) and multiplying by $i\bar{\mu}^{2\epsilon}/(4\pi)^2$, we conclude that the box classical term is given by

$$\mathcal{M}^{\Box(-3)}_{e^4,(p)} = 8e^4 Q_1^2 Q_2^2 \frac{1}{(4\pi)^2} \left(\frac{-q^2}{\bar{\mu}^2} \right)^{-\epsilon} \frac{\sqrt{\pi}(\bar{m}_1 + \bar{m}_2)}{\sqrt{-q^2}} \left(y - \frac{2\epsilon y^2}{y-1} \right) \frac{e^{\gamma_E \epsilon} \Gamma(\frac{1}{2} - \epsilon)^2 \Gamma(\frac{1}{2} + \epsilon)}{2\Gamma(1-2\epsilon)} \,. \tag{3.46}$$

The final diagram is the crossed box diagram

$$i\mathcal{M}^{\boxtimes}_{e^4}(\sigma, q^2) = \tag{3.47}$$



,

for which we find that

$$\mathcal{M}^{\boxtimes(-4)}_{e^4,(p)} = 0 \,, \tag{3.48}$$

$$\mathcal{M}^{\boxtimes(-3)}_{e^4,(p)} = 8e^4 Q_1^2 Q_2^2 \frac{1}{(4\pi)^2} \left(\frac{-q^2}{\bar{\mu}^2} \right)^{-\epsilon} \frac{\sqrt{\pi}(\bar{m}_1 + \bar{m}_2)}{\sqrt{-q^2}} \left(-y + \frac{2\epsilon y^2}{y+1} \right) \frac{e^{\gamma_E \epsilon} \Gamma(\frac{1}{2} - \epsilon)^2 \Gamma(\frac{1}{2} + \epsilon)}{2\Gamma(1-2\epsilon)} \,. \tag{3.49}$$

Combining all of our results for the triangle and (crossed-)box diagrams, the scalar QED one-loop amplitude in the potential region is given by

$$\begin{aligned} \mathcal{M}^{1-\text{loop}}_{e^4,(p)} = 8e^4 Q_1^2 Q_2^2 \frac{1}{(4\pi)^2} \left(\frac{-q^2}{\bar{\mu}^2} \right)^{-\epsilon} \Bigg[& \frac{1}{(-q^2)} \frac{i\pi \bar{m}_1 \bar{m}_2 y^2}{\sqrt{y^2-1}} \frac{e^{\gamma_E \epsilon} \Gamma(-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(-2\epsilon)} \\ & + \frac{1}{\sqrt{-q^2}} \sqrt{\pi}(\bar{m}_1 + \bar{m}_2) \left(1 - \frac{4\epsilon y^2}{y^2-1} \right) \frac{e^{\gamma_E \epsilon} \Gamma(\frac{1}{2} - \epsilon)^2 \Gamma(\frac{1}{2} + \epsilon)}{2\Gamma(1-2\epsilon)} \Bigg] \,. \end{aligned} \tag{3.50}$$

3.4 One-loop Amplitudes in Einstein-Maxwell Theory

For the computation of the one-loop scattering amplitudes in gravity, which include diagrams with graviton propagators only and diagrams with both graviton and photon propagators, we employ the numerical unitarity method [110–113] using the Caravel [114] framework. The numerical evaluations performed will give numerical values of the integral coefficients associated with a basis of master integrals. Analytic expressions are then determined using a procedure known as functional reconstruction [115, 116].

The one-loop amplitude in pure gravity in the potential region is given by

$$\begin{aligned}\mathcal{M}^{1-\text{loop}}_{G^2,(p)} = 512\pi^2 G^2 \bar{m}_1^2 \bar{m}_2^2 \frac{1}{(4\pi)^2}\left(\frac{-q^2}{\bar{\mu}^2}\right)^{-\epsilon}\Bigg[&\frac{1}{(-q^2)}\frac{i\pi\bar{m}_1\bar{m}_2(1-2y^2)^2}{4\sqrt{y^2-1}}\frac{e^{\gamma_E\epsilon}\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(-2\epsilon)}\\ &-\frac{1}{\sqrt{-q^2}}\frac{3\sqrt{\pi}}{8}(\bar{m}_1+\bar{m}_2)(1-5y^2)\frac{e^{\gamma_E\epsilon}\Gamma(\frac{1}{2}-\epsilon)^2\Gamma(\frac{1}{2}+\epsilon)}{2\Gamma(1-2\epsilon)}\Bigg],\end{aligned} \tag{3.51}$$

and the result for the one-loop amplitude in Einstein-Maxwell is

$$\begin{aligned}\mathcal{M}^{1-\text{loop}}_{Ge^2,(p)} = 64\pi G e^2 \bar{m}_1 \bar{m}_2 \frac{1}{(4\pi)^2}\left(\frac{-q^2}{\bar{\mu}^2}\right)^{-\epsilon}\Bigg[&\frac{1}{(-q^2)}\frac{i\pi Q_1 Q_2 \bar{m}_1\bar{m}_2 y(1-2y^2)}{\sqrt{y^2-1}}\frac{e^{\gamma_E\epsilon}\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(-2\epsilon)}\\ &+\frac{1}{\sqrt{-q^2}}\frac{\sqrt{\pi}}{4}\Big((Q_1^2\bar{m}_2+Q_2^2\bar{m}_1)(1-3y^2)-12Q_1Q_2(\bar{m}_1+\bar{m}_2)y\Big)\frac{e^{\gamma_E\epsilon}\Gamma(\frac{1}{2}-\epsilon)^2\Gamma(\frac{1}{2}+\epsilon)}{2\Gamma(1-2\epsilon)}\Bigg].\end{aligned} \tag{3.52}$$

The 2PM scattering amplitude in Einstein-Maxwell theory in the potential region is then given by the sum

$$\mathcal{M}_2 = \mathcal{M}^{1-\text{loop}}_{G^2,(p)} + \mathcal{M}^{1-\text{loop}}_{Ge^2,(p)} + \mathcal{M}^{1-\text{loop}}_{e^4,(p)}\,. \tag{3.53}$$

4 Effective Field Theory

4.1 Lagrangian Description

Having calculated the 2PM scattering amplitude in Einstein-Maxwell theory, we will now extract the effective Hamiltonian describing the classical dynamics of a compact binary system by matching the full theory amplitudes to an EFT. We follow the EFT framework of Ref. [59], which describes two nonrelativistic massive scalar fields $\phi_{1,2}$ interacting through a non-local in space potential. In the center-of-mass frame, the Lagrangian of the system is given by

$$\begin{aligned}\mathcal{L} = &\sum_{i=1}^{2}\int\frac{\mathrm{d}^{D-1}\boldsymbol{k}}{(2\pi)^{D-1}}\phi_i^\dagger(-\boldsymbol{k})\left(i\partial_t-\sqrt{\boldsymbol{k}^2+m_i^2}\right)\phi_i(\boldsymbol{k})\\ &-\int\frac{\mathrm{d}^{D-1}\boldsymbol{k}}{(2\pi)^{D-1}}\frac{\mathrm{d}^{D-1}\boldsymbol{k}'}{(2\pi)^{D-1}}V(\boldsymbol{k},\boldsymbol{k}')\phi_1^\dagger(\boldsymbol{k}')\phi_1(\boldsymbol{k})\phi_2^\dagger(-\boldsymbol{k}')\phi_2(-\boldsymbol{k})\,.\end{aligned} \tag{4.1}$$

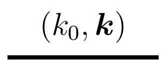
This EFT is obtained by integrating out the off-shell potential graviton and photon modes from the full theory and taking the nonrelativistic limit, so that anti-particles are absent. The form of the potential in momentum space is given by the following ansatz[5] [102]

$$
\begin{aligned}
V(\boldsymbol{k},\boldsymbol{k}') &= \sum_{n=1}^{\infty} \frac{(1/2)^n (4\pi)^{(D-1)/2}}{|\boldsymbol{k}-\boldsymbol{k}'|^{D-1-n}} \frac{\Gamma[(D-1-n)/2]}{\Gamma(n/2)} c_n\left(\frac{\boldsymbol{k}^2+\boldsymbol{k}'^2}{2}\right) \\
&= \frac{4\pi}{|\boldsymbol{k}-\boldsymbol{k}'|^2} c_1\left(\frac{\boldsymbol{k}^2+\boldsymbol{k}'^2}{2}\right) + \frac{2\pi^2}{|\boldsymbol{k}-\boldsymbol{k}'|} c_2\left(\frac{\boldsymbol{k}^2+\boldsymbol{k}'^2}{2}\right) + \cdots ,
\end{aligned}
\tag{4.2}
$$

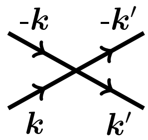
where the c_n are free coefficients that are fixed by EFT matching and encode the conservative dynamics, and n labels the PM order at which c_n contributes.

4.2 Scattering Amplitudes

From Eq. (4.1) we can derive the Feynman rules of the EFT,



$$= \frac{i}{k_0 - \sqrt{\boldsymbol{k}^2 + m_i^2} + i\varepsilon} ,$$

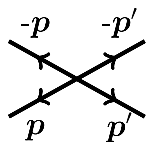


$$= -iV(\boldsymbol{k},\boldsymbol{k}') , \tag{4.3}$$

and subsequently construct the EFT amplitudes. The topology of the EFT amplitudes is given by iterated bubbles due to the absence of anti-particles. At 2PM order, we only need to consider two diagrams: the contact term in Eq. (4.3), and the diagram with a single bubble. Recall that we work in an all-outgoing momenta convention.

For the EFT scattering amplitude at 1PM order,

$$i\mathcal{M}_1^{\mathrm{EFT}} =$$



$$= -iV(-\boldsymbol{p},\boldsymbol{p}') . \tag{4.4}$$

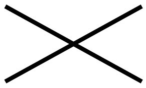
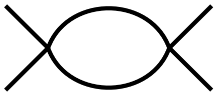
only the first term of the series in Eq. (4.2) contributes. The result is given by

$$\mathcal{M}_1^{\mathrm{EFT}} = -\frac{4\pi}{(-q^2)} c_1(\boldsymbol{p}^2) , \tag{4.5}$$

where we substituted $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{p}'$ for the momentum transfer, interchanged $\boldsymbol{q}^2$ with $(-q^2)$ since the momentum transfer is purely spatial in the center of mass frame, and used the condition $\boldsymbol{p}^2 = \boldsymbol{p}'^2$ valid for elastic scattering.

[5] In contrast to Ref. [102], here we absorb the coupling constant inside the c_n coefficients since we have two couplings in the theory, α and G

The EFT amplitude at 2PM order is given by a sum of two diagrams

$i\mathcal{M}_2^{\rm EFT} =$  $+$  , (4.6)

where we now substitute the second term in the potential for the contact interaction, but substitute the first term in the potential for the vertices in the bubble diagram. We get

$$\begin{aligned}\mathcal{M}_2^{\rm EFT} =& -\frac{2\pi^2}{\sqrt{-q^2}}c_2(\boldsymbol{p}^2)\\ &+32\pi^2\frac{1}{(4\pi)^2}\left(\frac{-q^2}{\bar{\mu}^2}\right)^{-\epsilon}\frac{1}{(-q^2)}\frac{i\pi E_1E_2}{\bar{m}_1\bar{m}_2\sqrt{y^2-1}}c_1^2\frac{e^{\gamma_E\epsilon}\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(-2\epsilon)}\\ &+32\pi^2\frac{1}{(4\pi)^2}\left(\frac{-q^2}{\bar{\mu}^2}\right)^{-\epsilon}\frac{\sqrt{\pi}}{\sqrt{-q^2}}\left[\left\{-\epsilon\frac{2E_1E_2(E_1+E_2)}{\bar{m}_1^2\bar{m}_2^2(y^2-1)}+\frac{E_1^2-E_1E_2+E_2^2}{2E_1E_2(E_1+E_2)}\right\}c_1^2\right.\\ &\qquad\qquad\left.+\frac{2E_1E_2}{E_1+E_2}c_1c_1'\right]\frac{e^{\gamma_E\epsilon}\Gamma(\frac{1}{2}-\epsilon)^2\Gamma(\frac{1}{2}+\epsilon)}{\Gamma(1-2\epsilon)}+\cdots\,,\end{aligned} \tag{4.7}$$

where in the above c_1 is a function of $\bar{\boldsymbol{p}}^2 = \boldsymbol{p}^2 - \boldsymbol{q}^2/4$ since for the bubble diagram we expanded in the soft region in order to facilitate the extraction of the superclassical and classical terms[6]; c_1' denotes the first derivative of c_1 with respect to $\bar{\boldsymbol{p}}^2$, and the ellipsis stand for higher-order quantum corrections which are irrelevant for the classical physics.

4.3 EFT Matching and Conservative two-body Hamiltonian

The EFT in Eq. (4.1) reproduces the same classical physics as the full theory in the potential region. Thus, we require

$$\mathcal{M}_n = 4E_1E_2\mathcal{M}_n^{\rm EFT}\,, \tag{4.8}$$

at every order in the PM expansion. In the above, we divided the full theory scattering amplitudes by the nonrelativistic normalization factor of $4E_1E_2$ in order to perform the EFT matching. Then, the coefficients in the potential are determined to be

$$c_1 = \frac{Gm_1^2m_2^2}{E_1E_2}(1-2\sigma^2)+\frac{e^2Q_1Q_2}{4\pi E_1E_2}(m_1m_2\sigma)\,, \tag{4.9}$$

$$c_2 = G^2\left[\frac{3m_1^2m_2^2(m_1+m_2)(1-5\sigma^2)}{4E_1E_2}-\frac{4m_1^3m_2^3\sigma(1-2\sigma^2)(E_1+E_2)}{E_1^2E_2^2}\right. \tag{4.10}$$

[6]The c_n coefficients will scale as $\hbar^{-n}$ when restoring factors of $\hbar$.

$$
\left. - \frac{m_1^4 m_2^4 (1-2\sigma^2)^2 (E_1^2 + E_1 E_2 + E_2^2)}{2E_1^3 E_2^3 (E_1+E_2)} \right]
$$

$$
+ \frac{Ge^2}{4\pi} \left[\frac{3Q_1 Q_2 m_1 m_2 (m_1+m_2)\sigma}{E_1 E_2} - \frac{m_1 m_2 (Q_1^2 m_2 + Q_2^2 m_1)(1-3\sigma^2)}{4E_1E_2} \right.
$$

$$
\left. + \frac{Q_1 Q_2 m_1^2 m_2^2 (1-6\sigma^2)(E_1+E_2)}{E_1^2 E_2^2} - \frac{Q_1 Q_2 m_1^3 m_2^3 \sigma (1-2\sigma^2)(E_1^2 + E_1E_2 + E_2^2)}{E_1^3 E_2^3 (E_1+E_2)} \right]
$$

$$
+ \frac{e^4 Q_1^2 Q_2^2}{16\pi^2} \left[- \frac{m_1+m_2}{2E_1E_2} + \frac{m_1 m_2 \sigma (E_1+E_2)}{E_1^2 E_2^2} - \frac{m_1^2 m_2^2 \sigma^2 (E_1^2 + E_1 E_2 + E_2^2)}{2E_1^3 E_2^3 (E_1+E_2)} \right].
$$

We note than when we substituted the results of Eqs. (3.50)-(3.52) for the 2PM scattering amplitude in Eq. (4.8) we replaced the soft variables $(\bar{m}_i, y)$ with the original variables (m_i, σ) since they differ by $\mathcal{O}(q^2)$ corrections, as can be seen from Eqs. (2.9) and (2.13), which are quantum. The superclassical terms present in the full theory and EFT canceled each other in the matching procedure, as we expected. Also, we set the dimensional regulator ϵ equal to zero for the classical terms in the full theory and EFT since these terms are finite.

The conservative two-body potential in position space is then obtained by plugging the above expressions for the c_n coefficients back in Eq. (4.2) and taking the Fourier transform of the momentum transfer $|\boldsymbol{k} - \boldsymbol{k}'|$ and putting the external legs on-shell, $\boldsymbol{k}^2 = \boldsymbol{k}'^2 = \boldsymbol{p}^2$. The Hamiltonian for the system of two charged, massive spinless particles is then

$$
H(\boldsymbol{p}, \boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + \frac{c_1(\boldsymbol{p}^2)}{|\boldsymbol{r}|} + \frac{c_2(\boldsymbol{p}^2)}{|\boldsymbol{r}|^2} + \cdots . \tag{4.11}
$$

4.4 Scattering Angle

The scattering angle can be obtained from our Hamiltonian (4.11) by solving the equations of motion. In Ref. [63], the scattering angle was expressed as a perturbative series in inverse powers of the angular momentum $J = b|\boldsymbol{p}|$, where b is the impact parameter. At 2PM order, the scattering angle in the center-of-mass frames is given by

$$
2\pi\chi = \frac{d_1}{J} + \frac{d_2}{J^2}, \tag{4.12}
$$

where the d_i coefficients are defined in terms of the IR finite parts of the nonrelativistically normalized PM amplitudes we computed in Section 3, which we will denote as $\mathcal{M}_i'$. These are defined to be

$$
d_1 = \frac{E_1 E_2}{E} \frac{\boldsymbol{q}^2}{|\boldsymbol{p}|} \mathcal{M}_1', \qquad d_2 = \frac{E_1 E_2}{E} |\boldsymbol{q}| \mathcal{M}_2'. \tag{4.13}
$$

For Einstein-Maxwell theory, we have

$$\mathcal{M}'_1 = -\frac{4\pi G m_1^2 m_2^2}{E_1 E_2 \boldsymbol{q^2}}(1-2\sigma^2) - \frac{e^2 Q_1 Q_2}{E_1 E_2 \boldsymbol{q^2}}(m_1 m_2 \sigma)\,, \tag{4.14}$$

$$\begin{aligned}\mathcal{M}'_2 = &\frac{3\pi^2 G^2 m_1^2 m_2^2 (m_1+m_2)}{2E_1E_2|\boldsymbol{q}|}(5\sigma^2-1) + \frac{e^4 Q_1^2 Q_2^2 (m_1+m_2)}{16 E_1 E_2 |\boldsymbol{q}|} \\ &+ \frac{\pi G e^2 m_1 m_2}{8E_1E_2|\boldsymbol{q}|}\Big[(Q_1^2 m_2 + Q_2^2 m_1)(1-3\sigma^2) - 12 Q_1 Q_2 (m_1+m_2)\sigma\Big]\,,\end{aligned} \tag{4.15}$$

so the scattering angle is

$$\chi^{1\text{PM}} = \frac{2(2\sigma^2-1)}{\sqrt{\sigma^2-1}}\frac{Gm_1m_2}{J} - \frac{2\sigma}{\sqrt{\sigma^2-1}}\frac{e^2}{4\pi}\frac{Q_1Q_2}{J}\,, \tag{4.16}$$

$$\begin{aligned}\chi^{2\text{PM}} = &\frac{3\pi}{4}\frac{5\sigma^2-1}{\sqrt{1+2\nu(\sigma-1)}}\frac{G^2m_1^2m_2^2}{J^2} + \frac{1}{\sqrt{1+2\nu(\sigma-1)}}\frac{e^4}{32\pi}\frac{Q_1^2Q_2^2}{J^2} \\ &+ \frac{m_1 m_2}{\sqrt{1+2\nu(\sigma-1)}}\left[\frac{(Q_1^2m_2+Q_2^2m_1)(1-3\sigma^2) - 12Q_1Q_2(m_1+m_2)\sigma}{16(m_1+m_2)}\right]\frac{Ge^2}{J^2}\,,\end{aligned} \tag{4.17}$$

where $\nu = m_1 m_2/(m_1+m_2)^2$. Our result for the 2PM angle is in precise agreement with the result of Ref. [95].

5 Project Plan

5.1 Short-term Plan

In the previous sections we have shown the progress we made so far in the program that uses scattering amplitudes and effective field theory to solve the two-body problem in general relativity. Specifically, we focused on studying a binary system of charged, spinless compact bodies and we derived the conservative two-body Hamiltonian of the system.

Our aim for the present work is to use it as a basis for a pedagogical, introductory paper that is accessible for students new to this field. As such, we want to provide as much details as possible for the central ideas and results. To this end, we will:

- Perform consistency checks of our result against known results available in the literature, which includes the 2PL[7] Hamiltonian for scalar QED [117], the 2PM Hamiltonian in Einstein gravity [63], and the Hamiltonian for a charged point particle in a Reissner-Nordström background.

- In Section 3, add the relevant Feynman diagrams which give rise to the scattering amplitudes in Eqs. (3.51) and (3.52).

[7]The post-Lorentzian (PL) expansion is an expansion in powers of the electromagnetic coupling constant alone, including contributions to all orders in velocity.

- In Section 4, expand on the calculations for the EFT amplitudes and matching procedure and organize them more concisely, and (2) elaborate more on how to derive Eq. (4.12) for the scattering angle from the scattering amplitudes.

- Contrast the methodology we used in this work with other methods used in the literature such as generalized unitarity and double-copy constructions, and other approaches such as the worldline EFT formalism [118].

5.2 Four-year General Plan

First Stage

Duration: 5-6 months.
Objectives:

(a) Computation of the amplitude for classical scattering of gravitationally and electromagnetically interacting complex scalars at two-loop order.

(b) Calculation of gravitational observables at $\mathcal{O}(G^3)$, $\mathcal{O}(G^2\alpha)$, $\mathcal{O}(G\alpha^2)$, and $\mathcal{O}(\alpha^3)$.

Activities: Extension of our model for charged black holes — two massive, complex scalar fields coupled to Einstein-Maxwell theory — in the Caravel framework.
Deliverable: At least one published article in an indexed peer-reviewed journal.
Description:
As we go to higher orders in perturbation theory, one of the main obstacles to overcome is the evaluation of complicated Feynman integrals that appear in loop diagrams. To make progress, we have to resort to computer tools. In this regard, numerical approaches have proven quite useful, as evidenced by the one-loop amplitudes in Einstein-Maxwell theory we computed using the Caravel framework, which implements the numerical unitarity method. At present, the computation of two-loop amplitudes in Einstein-Maxwell theory can be automated in Caravel. However, in its current form, Caravel does not automate the expansion of the integrand in the soft and potential regions. Therefore, the extraction of the classical contributions to the scattering amplitude is not optimized in the current framework. To improve the scalability of the entire project to higher-loop orders, classical contributions have to be precisely identified and extracted within the Caravel framework so that quantum contributions can be truncated away, decreasing the computational complexity of the calculation. This will be the main focus of the first stage of the project. Once this is accomplished, and the relevant amplitude for classical scattering is obtained, the calculation of gravitational observables is straightforward.

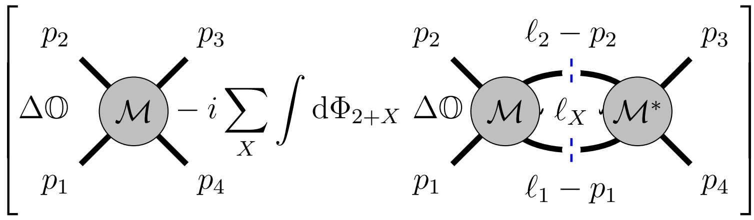
Second Stage

Duration: 1 year.
Objectives:

(a) Incorporate the effects of gravitational radiation on classical gravitational observables.

Activities: Implementation of the KMOC formalism into the Caravel framework.
Deliverable: At least one published article in an indexed peer-reviewed journal.
Description:
The scattering amplitude itself is not an observable, which is why the task of extracting the classical physics is not trivial. Classical gravitational observables can be computed directly using the KMOC formalism [62], which can account for radiative effects. In this formalism, an observable, ΔO, is defined as the change of an operator, $\mathbb{O}$, between the *in* and *out* states in a scattering process, i.e. $\Delta O = \langle \text{out}|\mathbb{O}|\text{out}\rangle - \langle \text{in}|\mathbb{O}|\text{in}\rangle$. In the classical limit, such an observable can be expressed diagrammatically as follows

$$\Delta O \sim i \int \mathrm{d}^D q \, \delta(-2p_1\cdot q)\delta(2p_2\cdot q)e^{ib\cdot q}$$



The observable we are after is specified by a *measurement function* $\Delta\mathbb{O}$ acting on the scattering amplitude $\mathcal{M}$ (first diagram), and on the amplitude to the left of a unitarity cut (second diagram). This measurement function translates to a numerator insertion or differential operator, neither operation currently available in Caravel. Extending Caravel to handle such operations constitutes the main focus of the second stage of the project.

Third Stage

Duration: 2-2.5 years.
Objectives:

(a) Computation of gravitational observables at three-loop order.

Activities: Extension of Caravel for the evaluation of gravitational observables in the KMOC formalism at the three-loop order.
Deliverable: At least one published article in an indexed peer-reviewed journal.
Description: The last part of the project will be dedicated to extending Caravel so that it can compute three-loop gravitational observables. At this loop order, potential *and* radiation modes give rise to conservative dynamics — a phenomenon known as the tail effect [119–121]. The distinction between conservative dynamics and radiation is then not as cleanly separated as in the two-loop order, i.e., radiation does not produce a purely dissipative effect. Additional loop diagrams and integration region have to be considered [65]. This part of the project is technically challenging, but once the computing infrastructure is up and running it will allow us to push the cutting-edge.

References

[1] A. Einstein, *Näherungsweise Integration der Feldgleichungen der Gravitation*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)* (1916) 688.

[2] R. A. Hulse and J. H. Taylor, *Discovery of a pulsar in a binary system*, *The Astrophysical Journal* **195** (1975) L51.

[3] J. H. Taylor and J. M. Weisberg, *A new test of general relativity - Gravitational radiation and the binary pulsar PSR 1913+16*, *The Astrophysical Journal* **253** (1982) 908.

[4] A. Einstein, *Über Gravitationswellen*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)* (1918) 154.

[5] LIGO Scientific, Virgo, B. P. Abbott et al., *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102 [1602.03837].

[6] LIGO Scientific, Virgo, B. P. Abbott et al., *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, *Phys. Rev. Lett.* **119** (2017) 161101 [1710.05832].

[7] LIGO Scientific, KAGRA, Virgo, R. Abbott et al., *Observation of Gravitational Waves from Two Neutron Star–Black Hole Coalescences*, *Astrophys. J. Lett.* **915** (2021) L5 [2106.15163].

[8] E. Capote et al., *Advanced LIGO detector performance in the fourth observing run*, *Physical Review D* **111** (2025) .

[9] KAGRA, T. Akutsu et al., *KAGRA: 2.5 Generation Interferometric Gravitational Wave Detector*, *Nature Astron.* **3** (2019) 35 [1811.08079].

[10] D. Reitze et al., *Cosmic Explorer: The U.S. Contribution to Gravitational-Wave Astronomy beyond LIGO*, *Bull. Am. Astron. Soc.* **51** (2019) 035 [1907.04833].

[11] M. Punturo et al., *The Einstein Telescope: a third-generation gravitational wave observatory*, *Class. Quant. Grav.* **27** (2010) 194002.

[12] M. Saleem et al., *The science case for LIGO-India*, *Class. Quant. Grav.* **39** (2022) 025004 [2105.01716].

[13] LISA Collaboration, P. Amaro-Seoane et al., *Laser Interferometer Space Antenna*, 1702.00786.

[14] N. Yunes and X. Siemens, *Gravitational-Wave Tests of General Relativity with Ground-Based Detectors and Pulsar Timing-Arrays*, *Living Rev. Rel.* **16** (2013) 9 [1304.3473].

[15] E. Berti et al., *Testing General Relativity with Present and Future Astrophysical Observations*, *Class. Quant. Grav.* **32** (2015) 243001 [1501.07274].

[16] LIGO Scientific, Virgo, B. P. Abbott et al., *Tests of general relativity with GW150914*, *Phys. Rev. Lett.* **116** (2016) 221101 [1602.03841].

[17] LIGO Scientific, Virgo, B. P. Abbott et al., *Tests of General Relativity with GW170817*, Phys. Rev. Lett. **123** (2019) 011102 [1811.00364].

[18] B. S. Sathyaprakash et al., *Extreme Gravity and Fundamental Physics*, 1903.09221.

[19] A. Bauswein and H. T. Janka, *Measuring neutron-star properties via gravitational waves from binary mergers*, Phys. Rev. Lett. **108** (2012) 011101 [1106.1616].

[20] K. Takami, L. Rezzolla and L. Baiotti, *Constraining the Equation of State of Neutron Stars from Binary Mergers*, Phys. Rev. Lett. **113** (2014) 091104 [1403.5672].

[21] L. Barack et al., *Black holes, gravitational waves and fundamental physics: a roadmap*, Class. Quant. Grav. **36** (2019) 143001 [1806.05195].

[22] B. Carr and F. Kuhnel, *Primordial Black Holes as Dark Matter: Recent Developments*, Ann. Rev. Nucl. Part. Sci. **70** (2020) 355 [2006.02838].

[23] B. S. Sathyaprakash, B. F. Schutz and C. Van Den Broeck, *Cosmography with the Einstein Telescope*, Class. Quant. Grav. **27** (2010) 215006 [0906.4151].

[24] J. M. Antelis and C. Moreno, *Obtaining gravitational waves from inspiral binary systems using LIGO data*, Eur. Phys. J. Plus **132** (2017) 10 [1610.03567].

[25] C. V. Vishveshwara, *Stability of the schwarzschild metric*, Phys. Rev. D **1** (1970) 2870.

[26] W. H. Press, *Long Wave Trains of Gravitational Waves from a Vibrating Black Hole*, Astrophys. J. Lett. **170** (1971) L105.

[27] M. Pürrer and C.-J. Haster, *Gravitational waveform accuracy requirements for future ground-based detectors*, Phys. Rev. Res. **2** (2020) 023151 [1912.10055].

[28] A. Buonanno and T. Damour, *Effective one-body approach to general relativistic two-body dynamics*, Phys. Rev. D **59** (1999) 084006 [gr-qc/9811091].

[29] A. Buonanno and T. Damour, *Transition from inspiral to plunge in binary black hole coalescences*, Phys. Rev. D **62** (2000) 064015 [gr-qc/0001013].

[30] Y. Mino, M. Sasaki and T. Tanaka, *Gravitational radiation reaction to a particle motion*, Phys. Rev. D **55** (1997) 3457 [gr-qc/9606018].

[31] T. C. Quinn and R. M. Wald, *An Axiomatic approach to electromagnetic and gravitational radiation reaction of particles in curved space-time*, Phys. Rev. D **56** (1997) 3381 [gr-qc/9610053].

[32] W. D. Goldberger and I. Z. Rothstein, *An Effective field theory of gravity for extended objects*, Phys. Rev. D **73** (2006) 104029 [hep-th/0409156].

[33] F. Pretorius, *Evolution of binary black hole spacetimes*, Phys. Rev. Lett. **95** (2005) 121101 [gr-qc/0507014].

[34] M. Campanelli, C. O. Lousto, P. Marronetti and Y. Zlochower, *Accurate evolutions of orbiting black-hole binaries without excision*, Phys. Rev. Lett. **96** (2006) 111101 [gr-qc/0511048].

[35] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz and J. van Meter, *Gravitational wave extraction from an inspiraling configuration of merging black holes*, *Phys. Rev. Lett.* **96** (2006) 111102 [gr-qc/0511103].

[36] S. A. Teukolsky, *Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations*, *Astrophys. J.* **185** (1973) 635.

[37] J. Droste, *The field of n moving centres in Einstein's theory of gravitation*, *Proc. Acad. Sci. Amst.* **19** (1916) 447.

[38] A. Einstein, L. Infeld and B. Hoffmann, *The Gravitational equations and the problem of motion*, *Annals Math.* **39** (1938) 65.

[39] A. Einstein and L. Infeld, *The Gravitational equations and the problem of motion. 2.*, *Annals Math.* **41** (1940) 455.

[40] T. Damour, P. Jaranowski and G. Schaefer, *Dynamical invariants for general relativistic two-body systems at the third postNewtonian approximation*, *Phys. Rev. D* **62** (2000) 044024 [gr-qc/9912092].

[41] T. Damour, P. Jaranowski and G. Schäfer, *Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems*, *Phys. Rev. D* **89** (2014) 064058 [1401.4548].

[42] P. Jaranowski and G. Schäfer, *Derivation of local-in-time fourth post-Newtonian ADM Hamiltonian for spinless compact binaries*, *Phys. Rev. D* **92** (2015) 124043 [1508.01016].

[43] B. Bertotti, *On gravitational motion*, *Nuovo Cim.* **4** (1956) 898.

[44] R. P. Kerr, *The Lorentz-covariant approximation method in general relativity I*, *Nuovo Cim.* **13** (1959) 469.

[45] B. Bertotti and J. Plebanski, *Theory of gravitational perturbations in the fast motion approximation*, *Annals Phys.* **11** (1960) 169.

[46] M. Portilla, *MOMENTUM AND ANGULAR MOMENTUM OF TWO GRAVITATING PARTICLES*, *J. Phys. A* **12** (1979) 1075.

[47] K. Westpfahl and M. Goller, *GRAVITATIONAL SCATTERING OF TWO RELATIVISTIC PARTICLES IN POSTLINEAR APPROXIMATION*, *Lett. Nuovo Cim.* **26** (1979) 573.

[48] M. Portilla, *SCATTERING OF TWO GRAVITATING PARTICLES: CLASSICAL APPROACH*, *J. Phys. A* **13** (1980) 3677.

[49] L. Bel, T. Damour, N. Deruelle, J. Ibanez and J. Martin, *Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity*, *Gen. Rel. Grav.* **13** (1981) 963.

[50] K. Westpfahl, *High-Speed Scattering of Charged and Uncharged Particles in General Relativity*, *Fortsch. Phys.* **33** (1985) 417.

[51] T. Damour, *Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory*, *Phys. Rev. D* **94** (2016) 104015 [1609.00354].

[52] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, *Testing binary dynamics in gravity at the sixth post-Newtonian level*, *Phys. Lett. B* **807** (2020) 135496 [2003.07145].

[53] D. Bini, T. Damour and A. Geralico, *Sixth post-Newtonian local-in-time dynamics of binary systems*, *Phys. Rev. D* **102** (2020) 024061 [2004.05407].

[54] D. Bini, T. Damour and A. Geralico, *Sixth post-Newtonian nonlocal-in-time dynamics of binary systems*, *Phys. Rev. D* **102** (2020) 084047 [2007.11239].

[55] S. Bini, S. Tiwari, Y. Xu, L. Smith, M. Ebersold, G. Principe et al., *Search for hyperbolic encounters of compact objects in the third LIGO-Virgo-KAGRA observing run*, *Phys. Rev. D* **109** (2024) 042009 [2311.06630].

[56] G. Kälin and R. A. Porto, *From Boundary Data to Bound States*, *JHEP* **01** (2020) 072 [1910.03008].

[57] G. Kälin and R. A. Porto, *From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)*, *JHEP* **02** (2020) 120 [1911.09130].

[58] G. Cho, G. Kälin and R. A. Porto, *From boundary data to bound states. Part III. Radiative effects*, *JHEP* **04** (2022) 154 [2112.03976].

[59] C. Cheung, I. Z. Rothstein and M. P. Solon, *From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion*, *Phys. Rev. Lett.* **121** (2018) 251101 [1808.02489].

[60] A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, *Post-Minkowskian Hamiltonians in general relativity*, *Phys. Rev. D* **100** (2019) 084040 [1906.01579].

[61] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, *General Relativity from Scattering Amplitudes*, *Phys. Rev. Lett.* **121** (2018) 171601 [1806.04920].

[62] D. A. Kosower, B. Maybee and D. O'Connell, *Amplitudes, Observables, and Classical Scattering*, *JHEP* **02** (2019) 137 [1811.10950].

[63] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order*, *Phys. Rev. Lett.* **122** (2019) 201603 [1901.04424].

[64] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon et al., *Scattering Amplitudes and Conservative Binary Dynamics at* $\mathcal{O}(G^4)$, *Phys. Rev. Lett.* **126** (2021) 171601 [2101.07254].

[65] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon et al., *Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at O(G4)*, *Phys. Rev. Lett.* **128** (2022) 161103 [2112.10750].

[66] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, 1308.1697.

[67] J. M. Henn and J. C. Plefka, *Scattering Amplitudes in Gauge Theories*, vol. 883. Springer, Berlin, 2014, 10.1007/978-3-642-54022-6.

[68] H. Kawai, D. C. Lewellen and S. H. H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, Nucl. Phys. B **269** (1986) 1.

[69] Z. Bern, J. J. M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, Phys. Rev. D **78** (2008) 085011 [0805.3993].

[70] Z. Bern, J. J. M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, Phys. Rev. Lett. **105** (2010) 061602 [1004.0476].

[71] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, Nucl. Phys. B **425** (1994) 217 [hep-ph/9403226].

[72] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, Nucl. Phys. B **435** (1995) 59 [hep-ph/9409265].

[73] Z. Bern and A. G. Morgan, *Massive loop amplitudes from unitarity*, Nucl. Phys. B **467** (1996) 479 [hep-ph/9511336].

[74] K. G. Chetyrkin and F. V. Tkachov, *Integration by parts: The algorithm to calculate β-functions in 4 loops*, Nucl. Phys. B **192** (1981) 159.

[75] A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, Phys. Lett. B **254** (1991) 158.

[76] Z. Bern, L. J. Dixon and D. A. Kosower, *Dimensionally regulated one loop integrals*, Phys. Lett. B **302** (1993) 299 [hep-ph/9212308].

[77] T. Gehrmann and E. Remiddi, *Differential equations for two-loop four-point functions*, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

[78] M. Driesse, G. U. Jakobsen, G. Mogull, J. Plefka, B. Sauer and J. Usovitsch, *Conservative Black Hole Scattering at Fifth Post-Minkowskian and First Self-Force Order*, Phys. Rev. Lett. **132** (2024) 241402 [2403.07781].

[79] B. Wardell, A. Pound, N. Warburton, J. Miller, L. Durkan and A. Le Tiec, *Gravitational Waveforms for Compact Binaries from Second-Order Self-Force Theory*, Phys. Rev. Lett. **130** (2023) 241402 [2112.12265].

[80] M. Khalil, A. Buonanno, J. Steinhoff and J. Vines, *Energetics and scattering of gravitational two-body systems at fourth post-Minkowskian order*, Phys. Rev. D **106** (2022) 024042 [2204.05047].

[81] Y. Iwasaki, *Fourth-order gravitational potential based on quantum field theory*, Lett. Nuovo Cim. **1S2** (1971) 783.

[82] Y. Iwasaki, *Quantum theory of gravitation vs. classical theory. - fourth-order potential*, Prog. Theor. Phys. **46** (1971) 1587.

[83] A. Buonanno, M. Khalil, D. O'Connell, R. Roiban, M. P. Solon and M. Zeng, *Snowmass White Paper: Gravitational Waves and Scattering Amplitudes*, in *Snowmass 2021*, 4, 2022, 2204.05194.

[84] M. Heusler, *Black Hole Uniqueness Theorems*, Cambridge Lecture Notes in Physics. Cambridge University Press, 1996.

[85] G. W. Gibbons, *Vacuum Polarization and the Spontaneous Loss of Charge by Black Holes*, *Commun. Math. Phys.* **44** (1975) 245.

[86] D. M. Eardley and W. H. Press, *Astrophysical processes near black holes*, *Ann. Rev. Astron. Astrophys.* **13** (1975) 381.

[87] R. D. Blandford and R. L. Znajek, *Electromagnetic extractions of energy from Kerr black holes*, *Mon. Not. Roy. Astron. Soc.* **179** (1977) 433.

[88] A. De Rujula, S. L. Glashow and U. Sarid, *CHARGED DARK MATTER*, *Nucl. Phys. B* **333** (1990) 173.

[89] M. L. Perl and E. R. Lee, *The search for elementary particles with fractional electric charge and the philosophy of speculative experiments*, *Am. J. Phys.* **65** (1997) 698.

[90] J. L. Feng, H. Tu and H.-B. Yu, *Thermal Relics in Hidden Sectors*, *JCAP* **10** (2008) 043 [0808.2318].

[91] J. L. Feng, M. Kaplinghat, H. Tu and H.-B. Yu, *Hidden Charged Dark Matter*, *JCAP* **07** (2009) 004 [0905.3039].

[92] V. Cardoso, C. F. B. Macedo, P. Pani and V. Ferrari, *Black holes and gravitational waves in models of minicharged dark matter*, *JCAP* **05** (2016) 054 [1604.07845].

[93] P. K. Gupta, *Binary dynamics from Einstein-Maxwell theory at second post-Newtonian order using effective field theory*, 2205.11591.

[94] C. R. T. Jones and M. Solon, *Scattering amplitudes and N-body post-Minkowskian Hamiltonians in general relativity and beyond*, *JHEP* **02** (2023) 105 [2208.02281].

[95] J. Wilson-Gerow, *Conservative scattering of Reissner-Nordström black holes at third post-Minkowskian order*, *JHEP* **05** (2024) 265 [2310.17731].

[96] J. Parra-Martinez, M. S. Ruf and M. Zeng, *Extremal black hole scattering at $\mathcal{O}(G^3)$: graviton dominance, eikonal exponentiation, and differential equations*, *JHEP* **11** (2020) 023 [2005.04236].

[97] P. V. Landshoff and J. C. Polkinghorne, *Iterations of regge cuts*, *Phys. Rev.* **181** (1969) 1989.

[98] C. Cheung and M. P. Solon, *Tidal Effects in the Post-Minkowskian Expansion*, *Phys. Rev. Lett.* **125** (2020) 191601 [2006.06665].

[99] Z. Bern, J. Parra-Martinez, R. Roiban, E. Sawyer and C.-H. Shen, *Leading Nonlinear Tidal Effects and Scattering Amplitudes*, *JHEP* **05** (2021) 188 [2010.08559].

[100] M. Beneke and V. A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, Nucl. Phys. B **522** (1998) 321 [hep-ph/9711391].

[101] V. A. Smirnov, *Evaluating Feynman integrals*, *Springer Tracts Mod. Phys.* **211** (2004) 1.

[102] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *Black Hole Binary Dynamics from the Double Copy and Effective Theory*, JHEP **10** (2019) 206 [1908.01493].

[103] J. F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, Phys. Rev. D **50** (1994) 3874 [gr-qc/9405057].

[104] Z. Bern, *Perturbative quantum gravity and its relation to gauge theory*, Living Rev. Rel. **5** (2002) 5 [gr-qc/0206071].

[105] S. Abreu, R. Britto and C. Duhr, *The SAGEX review on scattering amplitudes Chapter 3: Mathematical structures in Feynman integrals*, J. Phys. A **55** (2022) 443004 [2203.13014].

[106] G. 't Hooft and M. J. G. Veltman, *Regularization and Renormalization of Gauge Fields*, Nucl. Phys. B **44** (1972) 189.

[107] C. G. Bollini and J. J. Giambiagi, *Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter*, Nuovo Cim. B **12** (1972) 20.

[108] V. A. Smirnov, *Analytic tools for Feynman integrals*, vol. 250. Springer Tracts Mod. Phys., 2012, 10.1007/978-3-642-34886-0.

[109] B. R. Holstein and J. F. Donoghue, *Classical physics and quantum loops*, Phys. Rev. Lett. **93** (2004) 201602 [hep-th/0405239].

[110] H. Ita, *Two-loop Integrand Decomposition into Master Integrals and Surface Terms*, Phys. Rev. D **94** (2016) 116015 [1510.05626].

[111] S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier and B. Page, *Subleading Poles in the Numerical Unitarity Method at Two Loops*, Phys. Rev. D **95** (2017) 096011 [1703.05255].

[112] S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page and M. Zeng, *Two-Loop Four-Gluon Amplitudes from Numerical Unitarity*, Phys. Rev. Lett. **119** (2017) 142001 [1703.05273].

[113] S. Abreu, F. Febres Cordero, H. Ita, B. Page and M. Zeng, *Planar Two-Loop Five-Gluon Amplitudes from Numerical Unitarity*, Phys. Rev. D **97** (2018) 116014 [1712.03946].

[114] S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page et al., *Caravel: A C++ framework for the computation of multi-loop amplitudes with numerical unitarity*, Comput. Phys. Commun. **267** (2021) 108069 [2009.11957].

[115] A. von Manteuffel and R. M. Schabinger, *A novel approach to integration by parts reduction*, Phys. Lett. B **744** (2015) 101 [1406.4513].

[116] T. Peraro, *Scattering amplitudes over finite fields and multivariate functional reconstruction*, *JHEP* **12** (2016) 030 [1608.01902].

[117] Z. Bern, J. P. Gatica, E. Herrmann, A. Luna and M. Zeng, *Scalar QED as a toy model for higher-order effects in classical gravitational scattering*, *JHEP* **08** (2022) 131 [2112.12243].

[118] G. Kälin and R. A. Porto, *Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics*, *JHEP* **11** (2020) 106 [2006.01184].

[119] W. B. Bonnor, *Spherical gravitational waves*, *Philos. Trans. Roy. Soc. London A* **251** (1959) 233.

[120] L. Blanchet and T. Damour, *Tail Transported Temporal Correlations in the Dynamics of a Gravitating System*, *Phys. Rev. D* **37** (1988) 1410.

[121] L. Blanchet and T. Damour, *Hereditary effects in gravitational radiation*, *Phys. Rev. D* **46** (1992) 4304.